

# HOW TO CHOOSE WHAT YOU LIFT\*

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**Abstract.** We explore the lifting question in the context of cut-generating functions. Most of the prior literature on lifting for cut-generating functions focuses on which cut-generating functions have the unique lifting property. Here we develop a general theory for understanding how to do lifting for cut-generating functions which do not have the unique lifting property.

**Key words.** cutting plane theory, cut-generating functions, lattice free sets

**AMS subject classifications.** 90C10, 90C11

**1. Introduction.** We use  $\mathbb{N}$  to denote the set of natural numbers  $\{1, 2, \dots\}$ ,  $\mathbb{Z}$  to denote the set of integers,  $\mathbb{Z}_+$  to denote the set of nonnegative integers, and  $\mathbb{R}$  to denote the set of real numbers. For fixed  $n \in \mathbb{N}$ , let  $S$  be a closed subset of  $\mathbb{R}^n$  that does not contain the origin 0. In this manuscript, we consider subsets of the following form:

$$(1.1) \quad X_S(R, P) := \{(s, y) \in \mathbb{R}_+^k \times \mathbb{Z}_+^\ell : Rs + Py \in S\},$$

where  $k, \ell \in \mathbb{Z}_+$ ,  $R \in \mathbb{R}^{n \times k}$  and  $P \in \mathbb{R}^{n \times \ell}$  are matrices. We allow  $k = 0$  or  $\ell = 0$ , but not both.

A *cut-generating (function) pair*  $(\psi, \pi)$  for  $S$  is a pair of functions  $\psi, \pi: \mathbb{R}^n \rightarrow \mathbb{R}$  such that for every choice of  $k, \ell, R$  and  $P$ ,

$$(1.2) \quad \sum \psi(r)s_r + \sum \pi(p)y_p \geq 1$$

is an inequality separating 0 from  $\text{conv}(X_S(R, P))$  (sometimes cut-generating pairs are also called *valid* cut-generating functions, to emphasize that they give valid inequalities of the form (1.2) – we will also use this terminology sometimes to emphasize the validity). We use the convention in this manuscript that expressions of the form (1.2) have the first sum taken over the columns  $r$  of  $R$ , where  $s_r$  denotes the continuous variable associated with column  $r$ ; similarly, the second sum ranges over the columns  $p$  of  $P$ , and  $y_p$  denotes the integer variable associated with column  $p$ . The inequality (1.2) is known as a *cutting plane* or a *cut*. The literature studying the model (1.1) and cut-generating functions is extensive. We refer the reader to the surveys [22, 12, 4, 7, 8] and Chapter 6 of the textbook [14], and the references within, for an overview of the field.

There is a natural partial order on the set of valid pairs, namely  $(\psi', \pi') \leq (\psi, \pi)$  if and only if  $\psi' \leq \psi$  and  $\pi' \leq \pi$ . Since  $s$  and  $y$  are constrained to be nonnegative, whenever  $(\psi', \pi') \leq (\psi, \pi)$  all the cuts obtained from  $(\psi, \pi)$  are dominated by those obtained from  $(\psi', \pi')$ . The minimal elements under this partial order are called *minimal valid pairs*.

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The following approach has been instrumental in making cut-generating functions a computational tool for mixed-integer optimization. We introduce some important notions for this purpose. A set  $B \subseteq \mathbb{R}^n$  is called a *convex 0-neighborhood* if  $B$  is convex and  $0 \in \text{int}(B)$ . If  $B$  is a convex 0-neighborhood and  $S \cap \text{int}(B) = \emptyset$ , then  $B$  is called an  *$S$ -free convex 0-neighborhood*.  $B$  is called a *maximal  $S$ -free convex 0-neighborhood* if there does not exist a strict superset of  $B$  that is also an  $S$ -free convex 0-neighborhood. For any convex 0-neighborhood  $B$ , a function  $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$  is called a *representation of  $B$*  if  $B = \{r \in \mathbb{R}^n : \gamma(r) \leq 1\}$ . A convex neighborhood may have several representations (the classically known *gauge* function is one such representation). Representations of convex 0-neighborhoods was the main topic of study in [6, 11], where it was established that there always exists a *smallest* representation  $\gamma^*$  i.e.,  $\gamma^* \leq \gamma$  for all representations  $\gamma$  of  $B$ .

One creates cut-generating pairs by the following recipe:

1. Fix a maximal  $S$ -free convex 0-neighborhood  $B$ .
2. Let  $\gamma^*$  be the smallest representation of  $B$ .
3. Then  $\psi = \pi = \gamma^*$  will form a valid cut-generating pair.

However, the above recipe only produces “partially minimal” cut-generating pairs. One can show that for any other cut-generating pair  $(\psi', \pi') \leq (\psi, \pi)$ , one must have  $\psi' = \psi$ . But, there may exist other functions  $\pi' \leq \pi$  such that  $(\psi, \pi')$  is also a valid cut-generating pair. This motivates the following definition: For a given  $S$ , let  $B$  be a maximal  $S$ -free convex 0-neighborhood and let  $\psi$  be the smallest representation of  $B$ . Then  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}$  is called a *lifting of  $\psi$*  if  $(\psi, \pi)$  together forms a valid cut-generating pair (so  $\psi$  is a lifting of itself). The set of all liftings of  $\psi$  is partially ordered by pointwise dominance and one can thus define *minimal liftings*.

This approach to constructing cut-generating pairs is useful because for certain structured sets  $S$ , the smallest representations of maximal  $S$ -free convex 0-neighborhoods have nice, easy-to-compute “formulas”. Furthermore, for some classes of maximal  $S$ -free convex 0-neighborhoods, nice “formulas” exist for minimal liftings of the smallest representation. For a survey of these ideas, we refer to [4] and Section 6.3.4 in [14].

**Unique minimal liftings.** A central notion in the study of minimal liftings of a smallest representation  $\psi$  of  $B$  is the *extended lifting region*  $R(B)$  defined as:

$$(1.3) \quad R(B) := \{r \in \mathbb{R}^n : \pi_1(r) = \pi_2(r) \text{ for all minimal liftings } \pi_1, \pi_2 \text{ of } \psi\}.$$

From (1.3), if  $R(B) = \mathbb{R}^n$  then  $\psi$  has a unique minimal lifting. Moreover, one can derive nice “formulas” for this unique lifting in terms of the formula for  $\psi$ ; see Section 6 of the survey [4] for more details. A large class of maximal  $S$ -free convex 0-neighborhoods with this property have been identified and studied in many recent papers on minimal liftings [1, 2, 5, 9, 17]. However, the same literature shows that there are many sets  $B$  such that  $R(B) \subsetneq \mathbb{R}^n$ . The purpose of this manuscript is to describe minimal valid pairs that arise from such maximal  $S$ -free sets. Our work is motivated by Section 7 of [17], which initiated the study of this problem.

Consider  $p^* \in \mathbb{R}^n$  that does not fall into the extended lifting region. This means that there exist two minimal liftings of  $\psi$  that disagree on  $p^*$ . Dey and Wolsey studied the quantity  $V_\psi(p^*) := \inf\{\pi(p^*) : \pi \text{ minimal lifting of } \psi\}$  and showed that there exists a minimal lifting  $\pi$  such that  $\pi(p^*) = V_\psi(p^*)$  [17]; see Proposition 6 below for a statement that generalizes Dey and Wolsey’s arguments from  $\mathbb{R}^2$  and  $S = \mathbb{Z}^2$  to  $\mathbb{R}^n$ ,  $n \geq 2$  and general  $S \subseteq \mathbb{R}^n$ . Consider the collection of minimal liftings that equal  $V_\psi(p^*)$  on  $p^*$ .

DEFINITION 1. Let  $\mathcal{L}_{\psi, p^*}$  to be the set of all minimal liftings  $\pi$  of  $\psi$  such that  $\pi(p^*) = V_{\psi}(p^*)$ .

By definition of  $V_{\psi}(p^*)$  and the extended lifting region, all  $\pi \in \mathcal{L}_{\psi, p^*}$  agree on both  $V_{\psi}(p^*)$  and  $R(B)$ . Are there more values on which these liftings agree? Analogous to the extended lifting region, we define the *fixing region corresponding to  $p^*$*  to be the set of points on which all minimal liftings in  $\mathcal{L}_{\psi, p^*}$  agree.

**Statement of results.** We explore questions such as: What is a good description of the fixing region? How does the fixing region depend on  $p^*$ ? How much does the fixing region cover?

- Our first main result is Theorem 17, which provides an explicit inner approximation of the fixing region.
- The inner approximation we describe is used to show that for certain maximal  $S$ -free convex 0-neighborhoods  $B$  such that  $R(B) \subsetneq \mathbb{R}^n$ , there exists a  $p^*$  such that the fixing region is all of  $\mathbb{R}^n$ . In other words, after finding the optimal lifting coefficient  $V_{\psi}(p^*)$  for  $p^*$ , the lifting coefficients for all other vectors are uniquely determined for all minimal liftings that assign  $V_{\psi}(p^*)$  to the vector  $p^*$ . We say that such a set  $B$  is *one point fixable*. Proposition 27 shows that certain Type 3 triangles are one point fixable. As a corollary, in Proposition 29, we recover a result from [17] that Type 3 triangles resulting from the mixing set are one point fixable, using completely different and more geometric techniques. See [16, 20] for more on the mixing set.
- Theorem 21 says if our inner approximation of the fixing region shows that an  $S$ -free convex 0-neighborhood  $B$  is one point fixable, then the  $S + t$ -free convex 0-neighborhood  $B + t$  is also one point fixable for all vectors  $t \in \mathbb{R}^n$  such that  $B + t$  is still a 0-neighborhood. In other words, one point fixability is preserved under translations. These translation invariance results allow one to use these lifting arguments when one changes the basic feasible solution in a mixed-integer linear program, but wishes to use the same  $S$ -free neighborhood to derive cuts. A more detailed discussion of this point is done in the paper [9] and the survey [4].
- For our study, we develop a theory about so-called *partial cut-generating functions* – cut-generating functions that are only defined on a subset of  $\mathbb{R}^n$ . These results could be of general interest in the theory of cut-generating functions.
- Our work is very much geometric in flavor, complementing the algebraic approach taken by Dey and Wolsey in [17] who studied the problem in  $\mathbb{R}^2$ . Proposition 10 gives some evidence that the geometric approach may yield stronger liftings compared to the algebraic approach.

**2. Preliminaries and general facts about liftings.** We denote the columns of a matrix  $A$  by  $\text{col}(A)$ . For  $S \subseteq \mathbb{R}^n \setminus \{0\}$ , define:

$$(2.1) \quad \begin{aligned} W_S^+ &:= \{w \in \mathbb{R}^n : s + \lambda w \in S, \forall s \in S, \forall \lambda \in \mathbb{Z}_+\}, \\ W_S &:= \{w \in \mathbb{R}^n : s + \lambda w \in S, \forall s \in S, \forall \lambda \in \mathbb{Z}\}. \end{aligned}$$

Note that  $W_S = W_S^+ \cap (-W_S^+)$ .

Inspired by the recipe outlined in Section 1, we say a function  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$  is a *valid function for  $S$*  if  $(\psi, \psi)$  is a valid cut-generating pair for  $S$ . The recipe from Section 1 depends on the observation that representations (not necessarily smallest) of  $S$ -free

convex 0-neighborhoods (not necessarily maximal) are valid functions for  $S$ . However, not all valid functions of  $S$  are representations of  $S$ -free convex 0-neighborhoods. The notion of a lifting of  $\psi$  can be easily extended to any valid function  $\psi$  for  $S$ :  $\pi$  is a lifting of  $\psi$  if  $(\psi, \pi)$  forms a cut-generating pair for  $S$ . Under pointwise dominance, minimal elements of the set of liftings of a valid function  $\psi$  for  $S$  will be called minimal liftings of  $\psi$ .

**2.1. Partial cut-generating functions.** Let  $\mathcal{R}, \mathcal{P}$  be subsets of  $\mathbb{R}^n$  and let  $\psi : \mathcal{R} \rightarrow \mathbb{R}$  and  $\pi : \mathcal{P} \rightarrow \mathbb{R}$ . We define  $(\psi, \pi)$  to be a valid pair for  $S, \mathcal{R}, \mathcal{P}$ , if for every choice of  $k, \ell, R$  and  $P$  where the columns of  $R$  and  $P$  come from  $\mathcal{R}$  and  $\mathcal{P}$  respectively,

$$(2.2) \quad \sum \psi(r) s_r + \sum \pi(p) y_p \geq 1$$

is an inequality separating 0 from  $X_S(R, P)$ . The concept of a minimal valid pair is defined analogously to the case  $\mathcal{R} = \mathbb{R}^n, \mathcal{P} = \mathbb{R}^n$ . The notion of a valid function  $\psi : \mathcal{R} \rightarrow \mathbb{R}$  for  $S, \mathcal{R}$  is analogously defined. Let  $\mathcal{P} \subseteq \mathbb{R}^n$ , then we say  $\pi : \mathcal{P} \rightarrow \mathbb{R}$  is a lifting of a valid function  $\psi$  (valid for  $S, \mathcal{R}$ ), if  $(\psi, \pi)$  is a valid pair for  $S, \mathcal{R}, \mathcal{P}$ . The concept of a minimal lifting of  $\psi$  is analogously defined. When  $\psi$  and  $\pi$  are defined on strict subsets of  $\mathbb{R}^n$ , we refer to them as *partial cut-generating functions* and *partial cut-generating pairs*.

The following proposition establishes some results about partial cut-generating pairs. It is worth noting that setting  $\mathcal{R} = \mathcal{P} = \mathbb{R}^n$  in the following recovers the setting of cut-generating pairs.

**PROPOSITION 2.** *Let  $S \subseteq \mathbb{R}^n \setminus \{0\}$ ,  $\mathcal{R}, \mathcal{P}$  be subsets of  $\mathbb{R}^n$  and  $\psi : \mathcal{R} \rightarrow \mathbb{R}$  be a valid function for  $S, \mathcal{R}$ . Then the following hold:*

- (a) *For every minimal lifting  $\pi$  of  $\psi$ ,  $\pi(p) \leq \pi(p + w)$  for all  $p \in \mathcal{P}$  and  $w \in W_S^+$  such that  $p + w \in \mathcal{P}$ . Thus,  $\pi(p) = \pi(p + w)$  for all  $p \in \mathcal{P}$  and  $w \in W_S$  such that  $p + w \in \mathcal{P}$ .*
- (b) *Define  $\psi^* : \mathcal{R} \rightarrow \mathbb{R}$  as follows*

$$\psi^*(r) = \inf \{ \psi(r + w) : w \in W_S^+ \text{ such that } r + w \in \mathcal{R} \}$$

*Then  $(\psi, \psi^*)$  is a valid partial cut-generating pair for  $S, \mathcal{R}, \mathcal{R}$ .*

- (c) *If  $\mathcal{R} = \mathcal{P}$ , then every minimal lifting  $\pi$  of  $\psi$  satisfies  $\pi \leq \psi^*$ .*

*Proof.* We first establish that if  $\pi$  is any lifting of  $\psi$ , then  $\pi^* : \mathcal{P} \rightarrow \mathbb{R}$  defined as

$$\pi^*(p) := \inf_{w \in W_S^+} \left\{ \pi(p + w) : p + w \in \mathcal{P} \right\}$$

is also a lifting of  $\psi$ . Consider any  $R \in \mathcal{R}^k, P \in \mathcal{P}^\ell$  and  $(s, y) \in X_S(R, P)$ . Let  $r^i, i = 1, \dots, k$  be the columns of  $R$ . Let  $W \in \mathbb{R}^{n \times \ell}$  be any matrix whose columns are in  $W_S^+$  such that  $P + W \in \mathcal{P}^\ell$ . Let  $(\bar{s}, \bar{y})$  be constructed as follows:  $\bar{s} = s$  and  $\bar{y}_{p^j + w^j} = y_{p^j}$ , where  $p^j, w^j$  are the columns of  $P, W$  respectively for  $j = 1, \dots, \ell$ . Then  $R\bar{s} + (P + W)\bar{y} = Rs + Py + \bar{w}$  where  $\bar{w} \in W_S^+$  by definition of  $W$ . Thus,  $R\bar{s} + (P + W)\bar{y} \in S$ . Since  $(\psi, \pi)$  is a valid pair, we obtain

$$\sum_{i=1}^k \psi(r^i) \bar{s}_{r^i} + \sum_{j=1}^\ell \pi(p^j + w^j) \bar{y}_{p^j + w^j} \geq 1.$$

The above holds for all matrices  $W \in \mathbb{R}^{n \times \ell}$  whose columns are in  $W_S^+$  and  $P+W \in \mathcal{P}^\ell$ . Taking an infimum over all such  $W$  gives

$$\begin{aligned} \sum_{i=1}^k \psi(r^i) s_{r^i} + \sum_{j=1}^\ell \pi^*(p^j) y_{p^j} &= \sum_{i=1}^k \psi(r^i) s_{r^i} + \inf_W \left\{ \sum_{j=1}^\ell \pi(p^j + w^j) y_{p^j} \right\} \\ &= \inf_W \left\{ \sum_{i=1}^k \psi(r^i) s_{r^i} + \sum_{j=1}^\ell \pi(p^j + w^j) \bar{y}_{p^j + w^j} \right\} \geq 1. \quad \square \end{aligned}$$

From this we immediately obtain (a), since  $\pi^* \leq \pi$  for any minimal lifting  $\pi$  and also  $(\psi, \pi^*)$  is a valid pair. Thus by minimality of  $\pi$ ,  $\pi^* = \pi$  and so  $\pi(p) = \pi^*(p) \leq \pi(p+w)$  for all  $p \in \mathcal{P}$  and  $w \in W_S^+$  such that  $p+w \in \mathcal{P}$ .

Part (b) follows from the fact that  $\psi$  is a valid function for  $S$ .

For any minimal lifting  $\pi$ , we have  $\pi(r) \leq \psi(r)$  for all  $r \in \mathcal{R}(=\mathcal{P})$ . Using (a),  $\pi(p) \leq \pi(p+w) \leq \psi(p+w)$  for all  $p \in \mathcal{P}$  and  $w \in W_S^+$  such that  $p+w \in \mathcal{P}$ . Taking an infimum over all such  $w \in W_S^+$ , we obtain (c) (here we use  $\mathcal{R} = \mathcal{P}$ ).

The following result follows from standard calculations involving cut-generating functions and the proof is omitted [4].

**THEOREM 3.** *Let  $(\psi, \pi)$  be a minimal valid pair for  $S, \mathcal{R}, \mathcal{P}$ . Then  $\psi$  and  $\pi$  are both subadditive over  $\mathcal{R}$  and  $\mathcal{P}$  respectively, i.e.,  $\psi(r_1 + r_2) \leq \psi(r_1) + \psi(r_2)$  for all  $r_1, r_2 \in \mathcal{R}$  such that  $r_1 + r_2 \in \mathcal{R}$ , and  $\pi(p_1 + p_2) \leq \pi(p_1) + \pi(p_2)$  for all  $p_1, p_2 \in \mathcal{P}$  such that  $p_1 + p_2 \in \mathcal{P}$ . Moreover,  $\psi$  is positively homogeneous over  $\mathcal{R}$ , i.e., for all  $r \in \mathcal{R}$  and  $\lambda > 0$  such that  $\lambda r \in \mathcal{R}$ , we have  $\psi(\lambda r) = \lambda \psi(r)$ .*

When allowing partial cut-generating pairs to be defined on subsets of  $\mathbb{R}^n$ , it is natural to ask how such pairs behave when the domain is extended.

**QUESTION 4.** *Given  $\mathcal{R} \subseteq \mathcal{R}' \subseteq \mathbb{R}^n, \mathcal{P} \subseteq \mathcal{P}' \subseteq \mathbb{R}^n$ , and a valid pair  $(\psi, \pi)$  valid for  $S, \mathcal{R}, \mathcal{P}$ , does there always exist functions  $\psi', \pi'$  such that  $(\psi', \pi')$  is valid for  $S, \mathcal{R}', \mathcal{P}'$  and  $\psi', \pi'$  are extensions of  $\psi, \pi$ , i.e., they coincide on  $\mathcal{R}$  and  $\mathcal{P}$  respectively?*

The answer to Question 4 is NO in general. Indeed choosing  $\mathcal{R} = \emptyset$  and  $\mathcal{P} = \mathbb{R}^n$ , we obtain Gomory and Johnson's pure integer model, and we know that the discontinuous valid functions  $\pi$  for this model cannot be appended with any  $\psi$  to give a valid pair for the full mixed-integer model (see [15]). However, under certain conditions, such extensions can be constructed. Below,  $\text{cone}(X)$  denotes the conical hull of  $X \subseteq \mathbb{R}^n$ .

**THEOREM 5.** *Let  $\mathcal{R} \subseteq \mathcal{R}' \subseteq \mathbb{R}^n, \mathcal{P} \subseteq \mathcal{P}' \subseteq \mathbb{R}^n$ , and let  $(\psi, \pi)$  be a valid pair for  $S, \mathcal{R}, \mathcal{P}$ . Suppose  $\mathcal{R}', \mathcal{P}' \subseteq \text{cone}(\mathcal{R})$ . Then there exist functions  $\psi' : \mathcal{R}' \rightarrow \mathbb{R}, \pi' : \mathcal{P}' \rightarrow \mathbb{R}$  such that  $(\psi', \pi')$  is a minimal valid pair for  $S, \mathcal{R}', \mathcal{P}'$  and  $\psi', \pi'$  restricted to  $\mathcal{R}, \mathcal{P}$  dominate  $\psi, \pi$  respectively.*

*Proof.* For  $r' \in \mathcal{R}'$  define

$$v_\psi(r') := \inf \left\{ \sum_{r \in \mathcal{R}} \psi(r) h(r) : r' = \sum_{r \in \mathcal{R}} r h(r), \quad h : \mathcal{R} \rightarrow \mathbb{R}_+ \text{ finite support function} \right\}.$$

Similarly, for  $p' \in \mathcal{P}'$  define

$$\begin{aligned} v_\pi(p') &:= \inf \left\{ \sum_{r \in \mathcal{R}} \psi(r)h(r) + \sum_{p \in \mathcal{P}} \pi(p)g(p) : \right. \\ &\quad \left. p' = \sum_{r \in \mathcal{R}} rh(r) + \sum_{p \in \mathcal{P}} pg(p), \right. \\ &\quad \left. h : \mathcal{R} \rightarrow \mathbb{R}_+, g \in \mathcal{P} \rightarrow \mathbb{Z}_+, \text{ finite support functions} \right\}. \end{aligned}$$

Since  $\mathcal{R}', \mathcal{P}' \subseteq \text{cone}(\mathcal{R})$ , the infima defining  $v_\psi(r')$  and  $v_\pi(p')$  are over nonempty sets, so  $v_\psi(r')$  and  $v_\pi(p')$  are less than  $\infty$ .

Define functions  $\tilde{\psi} : \mathcal{R}' \rightarrow \mathbb{R}$  and  $\tilde{\pi} : \mathcal{P}' \rightarrow \mathbb{R}$  to be

$$\tilde{\psi}(r') := \begin{cases} v_\psi(r') & \text{if } v_\psi(r') > -\infty, \\ \psi(r') & \text{if } v_\psi(r') = -\infty \text{ and } r' \in \mathcal{R}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{\pi}(p') := \begin{cases} v_\pi(p') & \text{if } v_\pi(p') > -\infty, \\ \pi(p') & \text{if } v_\pi(p') = -\infty \text{ and } p' \in \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases}$$

From the comment above and the definition of  $\tilde{\psi}, \tilde{\pi}$ , both functions are well-defined. Moreover,  $\tilde{\psi}(r) \leq \psi(r)$  for  $r \in \mathcal{R}$  and  $\tilde{\pi}(p) \leq \pi(p)$  for  $p \in \mathcal{P}$ . Therefore  $(\tilde{\psi}, \tilde{\pi})$  dominates  $(\psi, \pi)$  on  $\mathcal{R}, \mathcal{P}$ . Since any valid pair is dominated by a minimal pair by a well-known application of Zorn's lemma, it is sufficient to show that  $(\tilde{\psi}, \tilde{\pi})$  is valid for  $S, \mathcal{R}', \mathcal{P}'$ .

Let  $R', P'$  be matrices with columns in  $\mathcal{R}', \mathcal{P}'$  respectively and consider  $(s', y')$  in  $X_S(R', P')$ . Let  $\varepsilon > 0$ . Consider any column  $r'$  of  $R'$ ; therefore,  $r' \in \mathcal{R}' \subseteq \text{cone}(\mathcal{R})$ . By the definition of  $\tilde{\psi}$ , there exists  $h_{r'} : \mathcal{R} \rightarrow \mathbb{R}_+$  so that  $h_{r'}$  has finite support and

$$(2.3) \quad r' = \sum_{r \in \mathcal{R}} rh_{r'}(r) \quad \text{and} \quad \tilde{\psi}(r') > \left( \sum_{r \in \mathcal{R}} \psi(r)h_{r'}(r) \right) - \varepsilon.$$

A similar argument shows that for each column  $p'$  of  $P'$ , there exists  $h_{p'} : \mathcal{R} \rightarrow \mathbb{R}_+$  and  $g_{p'} : \mathcal{P} \rightarrow \mathbb{Z}_+$ , both with finite support, satisfying

$$(2.4) \quad p' = \sum_{r \in \mathcal{R}} rh_{p'}(r) + \sum_{p \in \mathcal{P}} pg_{p'}(p) \quad \text{and} \quad \tilde{\pi}(p') > \left( \sum_{r \in \mathcal{R}} \psi(r)h_{p'}(r) + \sum_{p \in \mathcal{P}} \pi(p)g_{p'}(p) \right) - \varepsilon.$$

Define  $R$  to be matrix with columns that is the union of the supports of  $h_{r'}$  and  $g_{p'}$ , where  $r'$  is a column of  $R'$  and  $p'$  is a column of  $P'$ . Similarly, define  $P$  to be matrix with columns that is the union of the supports of  $g_{p'}$ , where  $p'$  is a column of  $P'$ . Define the following vectors  $(\tilde{s}, \tilde{y})$ , with coordinates indexed by the columns of  $R$  and  $P$  respectively:

$$\tilde{s}_r := \sum_{r' \in R'} h_{r'}(r)s'_{r'} + \sum_{p' \in P'} h_{p'}(r)y'_{p'} \quad \text{and} \quad \tilde{y}_p := \sum_{p' \in P'} g_{p'}(p)y'_{p'}$$

Notice  $(\tilde{s}, \tilde{y}) \in X_S(R, P)$  since  $R\tilde{s} + P\tilde{y} = R's' + P'y' \in S$  because  $(s', y') \in X_S(R', P')$ . Set  $M = \sum_{r' \in R'} s'_{r'} + \sum_{p' \in P'} y'_{p'}$ , which is a constant since  $s'$  and  $y'$  are fixed. As  $(\psi, \pi)$  is valid for  $S, \mathcal{R}, \mathcal{P}$ , we see that

$$\begin{aligned}
& \sum_{r' \in R'} \tilde{\psi}(r') s'_{r'} + \sum_{p' \in P'} \tilde{\pi}(p') y'_{p'} \\
& \geq \sum_{r' \in R'} \left[ \sum_{r \in \mathcal{R}} \psi(r) h_{r'}(r) - \varepsilon \right] s'_{r'} + \sum_{p' \in P'} \left[ \sum_{r \in \mathcal{R}} \psi(r) h_{p'}(r) + \sum_{p \in \mathcal{P}} \pi(p) g_{p'}(p) - \varepsilon \right] y'_{p'} \\
& = \sum_{\substack{r \in \mathcal{R} \\ r' \in R'}} \psi(r) h_{r'}(r) s'_{r'} + \sum_{\substack{r \in \mathcal{R} \\ p' \in P'}} \psi(r) h_{p'}(r) y'_{p'} + \sum_{\substack{p \in \mathcal{P} \\ p' \in P'}} \pi(p) g_{p'}(p) y'_{p'} - \varepsilon M \\
& = \sum_{r \in \mathcal{R}} \psi(r) \tilde{s}_r + \sum_{p \in \mathcal{P}} \pi(p) \tilde{y}_p - \varepsilon M \\
& \geq 1 - \varepsilon M.
\end{aligned}$$

Letting  $\varepsilon \rightarrow 0$  gives the desired result.  $\square$

**3. Fixing Region.** We now proceed to study minimal liftings of a valid function  $\psi$  (for some fixed closed set  $S \subseteq \mathbb{R}^n \setminus \{0\}$ ). For any  $p^* \in \mathbb{R}^n$ , recall the quantity  $V_\psi(p^*) := \inf\{\pi(p^*) : \pi \text{ minimal lifting of } \psi\}$  defined in the Introduction. Dey and Wolsey [17] provided an explicit formula:

$$(3.1) \quad V_\psi(p^*) = \sup_{w \in \mathbb{R}^n, N \in \mathbb{N}} \left\{ \frac{1 - \psi(w)}{N} : w + Np^* \in S \right\}.$$

Recall the set  $\mathcal{L}_{\psi, p^*}$  from Definition 1. Define the *fixing region* as

$$\mathcal{F}_{\psi, p^*} := \{p \in \mathbb{R}^n : \pi_1(p) = \pi_2(p) \text{ for all } \pi_1, \pi_2 \in \mathcal{L}_{\psi, p^*}\}.$$

In other words, the fixing region is the set of all points where all minimal liftings from  $\mathcal{L}_{\psi, p^*}$  take the same value.

**PROPOSITION 6.**  $\mathcal{L}_{\psi, p^*}$  is nonempty.

*Proof.* There are many ways to prove this; we do it via a particular construction from [17] that we will refer to later. Define

$$(3.2) \quad \phi(p) := \inf_{w \in \mathbb{R}^n, N \in \mathbb{N}} \left\{ \psi(w) + NV_\psi(p^*) : w + Np^* \in p + W_S \right\}. \quad \square$$

It was shown in [17] that  $\phi$  is a valid lifting of  $\psi$  and  $\phi(p^*) = V_\psi(p^*)$ .<sup>1</sup> Any minimal lifting  $\hat{\pi}$  dominating  $\phi$ , which exists by an application of Zorn's lemma, is in  $\mathcal{L}_{\psi, p^*}$ .

**4. Fixing Region for truncated affine lattices.** For the rest of the paper, we will specialize to sets  $S$  of the form  $S = (b + \mathbb{Z}^n) \cap P$  where  $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$  and  $P$  is a rational polyhedron. Such  $S$  were called *polyhedrally-truncated affine lattices* in [9]. For such  $S$ , maximal  $S$ -free convex 0-neighborhoods are all polyhedra that can

<sup>1</sup>Although [17] only deals with  $\mathbb{R}^2$  and  $S = \mathbb{Z}^2$ , the same proof works for general  $\mathbb{R}^n$  with  $n \geq 2$  and  $S \subseteq \mathbb{R}^n$ .

be written in the form  $B = \{r : a^i \cdot r \leq 1, i \in I\}$  where  $I$  is a finite set indexing the facets of  $B$  [3, 21]. For such a  $B$ , the smallest representation is given by

$$(4.1) \quad \psi(r) = \max_{i \in I} a^i \cdot r.$$

The value of  $V_\psi(p^*)$  can now be obtained geometrically in the following way. Define  $B(\lambda, p^*)$  as the translated cone in  $\mathbb{R}^n \times \mathbb{R}$  with  $\frac{1}{\lambda}(p^*, 1)$  as the apex and  $B \times \{0\}$  as the base, i.e.

$$(4.2) \quad B(\lambda, p^*) = \{(r, r_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : a^i \cdot r + (\lambda - a^i \cdot p^*)r_{n+1} \leq 1, i \in I\}.$$

The following was observed in [2]:

PROPOSITION 7.  $V_\psi(p^*) = \inf\{\lambda > 0 : B(\lambda, p^*) \text{ is } S \times \mathbb{Z}_+ \text{-free}\}.$

**4.1. A geometric perspective on  $\mathcal{L}_{\psi, p^*}$ .** The main tool for our geometric approach to understanding  $\mathcal{L}_{\psi, p^*}$  will be the polyhedron  $B(V_\psi(p^*), p^*)$  from (4.2).

DEFINITION 8. Let  $\psi$  be a valid function for  $S$  obtained from a maximal  $S$ -free 0-neighborhood  $B$  as in (4.1). Let  $p^* \in \mathbb{R}^n$ . A point  $(\bar{x}, \bar{x}_{n+1}) \in S \times \mathbb{Z}_+$  with  $\bar{x}_{n+1} \geq 1$  such that  $B(V_\psi(p^*), p^*)$  contains  $(\bar{x}, \bar{x}_{n+1})$  is called a blocking point for  $B(V_\psi(p^*), p^*)$ .

It was established in [2] that for every valid function  $\psi$  obtained from a maximal  $S$ -free 0-neighborhood  $B$  and every  $p^* \in \mathbb{R}^n$ , there exists a blocking point for  $B(V_\psi(p^*), p^*)$ . It is possible that there are more than one blocking point for a given  $B$  and  $p^*$ ; this fact will be exploited in later sections. The following lemma relates the algebraic formula (3.1) for  $V_\psi(p^*)$  and the important geometric notion of a blocking point for  $B(V_\psi(p^*), p^*)$ .

LEMMA 9. Suppose  $\psi$  is a valid function for  $S$  obtained from a maximal  $S$ -free 0-neighborhood  $B$  as in (4.1). If  $(\bar{x}, \bar{x}_{n+1}) \in S \times \mathbb{Z}_+$  is a blocking point for  $B(V_\psi(p^*), p^*)$ , then

$$(\bar{x} - \bar{x}_{n+1}p^*, \bar{x}_{n+1}) \in \operatorname{argmax}_{w \in \mathbb{R}^n, N \in \mathbb{N}} \left\{ \frac{1 - \psi(w)}{N} : w + Np^* \in S \right\}.$$

Conversely, if  $(w, N) \in \mathbb{R}^n \times \mathbb{N}$  is a maximizer for (3.1), then  $(w + Np^*, N)$  is a blocking point for  $B(V_\psi(p^*), p^*)$ .

*Proof.* From Equation (4.2),  $(\bar{x}, \bar{x}_{n+1})$  is a blocking point for  $B(V_\psi(p^*), p^*)$  if and only if  $a^i \cdot \bar{x} + (V_\psi(p^*) - a^i \cdot p^*)\bar{x}_{n+1} \leq 1$  for all  $i \in I$ , and there exists some  $i^* \in I$  such that  $a^{i^*} \cdot \bar{x} + (V_\psi(p^*) - a^{i^*} \cdot p^*)\bar{x}_{n+1} = 1$ . Rearranging these inequalities and equality shows that  $\bar{x}_{n+1}V_\psi(p^*) + \max_{i \in I} \{a^i \cdot (\bar{x} - \bar{x}_{n+1}p^*)\} = 1$ . Thus  $(\bar{x}, \bar{x}_{n+1})$  is a blocking point for  $B(V_\psi(p^*), p^*)$  if and only if  $V_\psi(p^*) = \frac{1 - \psi(\bar{x} - \bar{x}_{n+1}p^*)}{\bar{x}_{n+1}}$ . We are then done because of formula (3.1).  $\square$

Since blocking points always exist, Lemma 9 says that the supremum in (3.1) is actually a maximum.

**Algebra versus Geometry.** Although the function  $\phi$  defined in (3.2) may not be a minimal lifting for  $\psi$ , it gives an explicit formula for computing the lifting values. Our geometric perspective provides an alternative function, which we show in Proposition 10 always dominates the algebraic construction of (3.2). Let  $\psi_{B(\lambda, p^*)} : (\mathbb{R}^n \times \mathbb{R}) \rightarrow \mathbb{R}$  be defined from  $B(\lambda, p^*)$  using (4.1). Note that, by construction,  $\psi_{B(V_\psi(p^*), p^*)}(p^*, 1) = V_\psi(p^*)$  and  $\psi_{B(V_\psi(p^*), p^*)}(r, 0) = \psi(r)$  for all  $r \in \mathbb{R}^n$ .



Define

$$(4.3) \quad \psi_{B(V_\psi(p^*), p^*)}^*((r, r_{n+1})) := \inf_{(w, z) \in W_{S \times \mathbb{Z}_+}^+} \psi_{B(V_\psi(p^*), p^*)}((r, r_{n+1}) + (w, z)).$$

The function  $\pi(p) = \psi_{B(V_\psi(p^*), p^*)}^*(p, 0)$  can be shown to be a valid lifting of  $\psi$ . Notice that  $\psi_{B(V_\psi(p^*), p^*)}^*$  is the lifting defined in Proposition 2 for the function  $\psi_{B(V_\psi(p^*), p^*)}$ . It is well-known that  $\psi_{B(V_\psi(p^*), p^*)}^*$  is a subadditive function.

**PROPOSITION 10.** *Consider  $\phi$  and  $\psi_{B(V_\psi(p^*), p^*)}^*$  defined in Equations (3.2) and (4.3), respectively. Then  $\psi_{B(V_\psi(p^*), p^*)}^*(p, 0) \leq \phi(p)$  for all  $p \in \mathbb{R}^n$ .*

*Proof.* For any  $w \in \mathbb{R}^n$  and  $N \in \mathbb{N}$  such that  $w + Np^* = p + x$  for some  $x \in W_S$ , we have

$$\begin{aligned} \psi(w) + NV_\psi(p^*) &= \psi_{B(V_\psi(p^*), p^*)}((w, 0)) + N\psi_{B(V_\psi(p^*), p^*)}((p^*, 1)) \\ &\geq \psi_{B(V_\psi(p^*), p^*)}^*((w, 0)) + N\psi_{B(V_\psi(p^*), p^*)}^*((p^*, 0)) \\ &\geq \psi_{B(V_\psi(p^*), p^*)}^*((w, 0) + N(p^*, 0)) \\ &\geq \psi_{B(V_\psi(p^*), p^*)}^*((p, 0) + (x, 0)) \\ &= \psi_{B(V_\psi(p^*), p^*)}^*((p, 0)), \end{aligned}$$

where the second inequality follows from subadditivity of  $\psi_{B(V_\psi(p^*), p^*)}^*$ . Taking the infimum on the left gives the desired result.  $\square$

**A universal upper bound.** In answering what vectors lie in  $\mathcal{F}_{\psi, p^*}$ , we first show an upper bound on the value of minimal liftings and then show this upper bound is tight. Theorem 12 gives such an upper bound, stating that the restriction of  $\psi_{B(V_\psi(p^*), p^*)}^*$  to  $\mathbb{R}^n \times \{0\}$  is a universal upper bound for all minimal liftings  $\pi \in \mathcal{L}_{\psi, p^*}$ . The following technical lemma will be useful for this purpose.

**LEMMA 11.** *Let  $B = \{r : a^i \cdot r \leq 1 \mid i \in I\}$  be a polyhedron in  $\mathbb{R}^n$  containing 0 in its interior and let  $p^* \in \mathbb{R}^n$  and  $\lambda > 0$ . For  $(\bar{r}, \bar{r}_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+$  and  $\mu \geq 0$ , define  $r' = (\bar{r}, \bar{r}_{n+1}) - \mu(p^*, 1)$ . Then  $\psi_{B(\lambda, p^*)}((\bar{r}, \bar{r}_{n+1})) = \psi_{B(\lambda, p^*)}(r') + \mu\psi_{B(\lambda, p^*)}((p^*, 1))$ .*

*Proof.* We first show that

$$\begin{aligned} \operatorname{argmax}_{i \in I} \{a^i \cdot \bar{r} + (\lambda - a^i \cdot p^*)\bar{r}_{n+1}\} &= \operatorname{argmax}_{i \in I} \{a^i \cdot (\bar{r} - \bar{r}_{n+1}p^*)\} \\ &= \operatorname{argmax}_{i \in I} \{(a^i, (\lambda - a^i \cdot p^*)) \cdot r'\}. \end{aligned}$$

The first and second terms are equal since  $\lambda\bar{r}_{n+1}$  is a constant, while the first and the third terms are equal because for every  $i \in I$ ,

$$a^i \cdot \bar{r} + (\lambda - a^i \cdot p^*)\bar{r}_{n+1} = a^i \cdot (\bar{r} - \mu p^*) + (\lambda - a^i \cdot p^*)(\bar{r}_{n+1} - \mu) + \lambda\mu = (a^i, (\lambda - a^i \cdot p^*)) \cdot r' + \lambda\mu.$$

Let  $i^* \in \operatorname{argmax}_{i \in I} \{a^i \cdot \bar{r} + (\lambda - a^i \cdot p^*)\bar{r}_{n+1}\}$ , and so

$$\begin{aligned} \psi_{B(\lambda, p^*)}((\bar{r}, \bar{r}_{n+1})) &= a^{i^*} \cdot \bar{r} + (\lambda - a^{i^*} \cdot p^*)\bar{r}_{n+1} \\ &= (a^{i^*}, (\lambda - a^{i^*} \cdot p^*)) \cdot r' + (a^{i^*}, (\lambda - a^{i^*} \cdot p^*)) \cdot \mu(p^*, 1) \\ &= \psi_{B(\lambda, p^*)}(r') + \mu\psi_{B(\lambda, p^*)}((p^*, 1)). \end{aligned}$$

The last equality holds because  $(a^{i^*}, (\lambda - a^{i^*} \cdot p^*)) \cdot (p^*, 1) = \lambda = \psi_{B(\lambda, p^*)}((p^*, 1))$ .  $\square$

**THEOREM 12.** *Let  $\psi$  be a valid function for  $S$  obtained from a maximal  $S$ -free 0-neighborhood  $B$  as in (4.1). Let  $p^* \in \mathbb{R}^n$ . Let  $\psi_{B(V_\psi(p^*), p^*)}^*$  be obtained from  $\psi_{B(V_\psi(p^*), p^*)}$  as in Proposition 2. Then for every  $\pi \in \mathcal{L}_{\psi, p^*}$  and  $p \in \mathbb{R}^n$ ,  $\pi(p) \leq \psi_{B(V_\psi(p^*), p^*)}^*(p, 0)$ .*

*Proof.* Starting from  $\psi$  and  $\pi \in \mathcal{L}_{\psi, p^*}$ , we would like to apply Theorem 5 by extending from  $\mathcal{R} = \mathcal{P} = \mathbb{R}^n \times \{0\}$  (which can be thought of as the domain of  $\psi, \pi$ ) to  $\mathcal{R}' = \mathcal{P}' = \mathbb{R}^n \times \mathbb{R}_+$ , which is the domain of  $\psi_{B(V_\psi(p^*), p^*)}^*$ . However,  $\mathbb{R}^n \times \mathbb{R}_+ \not\subseteq \text{cone}(\mathbb{R}^n \times \{0\})$  and so the hypotheses of Theorem 5 are not satisfied. Instead, we will create related functions  $\hat{\psi}$  and  $\hat{\pi}$  in  $n + 1$  dimensions for which we can employ Theorem 5. This application of Theorem 5 will yield a minimal pair  $(\psi', \pi')$  in  $n + 1$  dimensions that matches  $(\psi, \pi)$  on the  $n$ -dimensional restricted space, but also dominates  $\psi_{B(V_\psi(p^*), p^*)}^*$  in  $n + 1$  dimensions.

First, define  $\mathcal{R} := (\mathbb{R}^n \times \{0\}) \cup \{(p^*, 1)\} \subseteq \mathbb{R}^n \times \mathbb{R}_+$  and  $\mathcal{P} := \mathbb{R}^n \times \{0\}$ . Define  $\hat{\psi} : \mathcal{R} \rightarrow \mathbb{R}$  by  $\hat{\psi}((r, 0)) = \psi(r)$  for all  $r \in \mathbb{R}^n$  and  $\hat{\psi}((p^*, 1)) = V_\psi(p^*)$ . Define  $\hat{\pi} : \mathcal{P} \rightarrow \mathbb{R}$  as  $\hat{\pi}((p, 0)) = \pi(p)$ .

**CLAIM 13.**  *$(\hat{\psi}, \hat{\pi})$  is valid for  $(S \times \mathbb{Z}_+), \mathcal{R}, \mathcal{P}$ .*

*Proof of Claim 13.* Consider any matrices  $R \in \mathbb{R}^{n \times k}$  and  $P \in \mathbb{R}^{n \times \ell}$  with columns in  $\mathcal{R}, \mathcal{P}$  respectively and let  $(\bar{s}, \bar{y}) \in X_{S \times \mathbb{Z}_+}(R, P)$ . If  $R$  does not contain  $(p^*, 1)$  or  $\bar{s}_{(p^*, 1)} = 0$ , we are done by the validity of  $(\psi, \pi)$ . Otherwise, since  $R\bar{s} + P\bar{y} \in S \times \mathbb{Z}_+$  and  $\mathcal{P} \subseteq \mathbb{R}^n \times \{0\}$ , we must have  $\bar{s}_{(p^*, 1)} \in \mathbb{Z}_+$ . Define  $\tilde{R} \in \mathbb{R}^{n \times k}$  to be the matrix with columns in  $\mathbb{R}^n$  that arise by truncating the columns of  $R \setminus \{(p^*, 1)\}$  to the first  $n$  coordinates. Define  $\tilde{P} \in \mathbb{R}^{n \times \ell+1}$  to be the matrix with columns in  $\mathbb{R}^n$  that arise by truncating the columns of  $P$  to the first  $n$  coordinates, and adding the column  $p^*$ .

Consider the pair  $(\tilde{s}, \tilde{y}) \in \mathbb{R}^k \times \mathbb{R}^{\ell+1}$  defined by  $\tilde{s}_r = \bar{s}_{(r, 0)}$  and  $\tilde{y}_p = \bar{y}_{(p, 0)}$  if  $p \neq p^*$  and  $\tilde{y}_{p^*} = \bar{y}_{(p^*, 0)} + \bar{s}_{(p^*, 1)}$ . Observe that  $\tilde{R}\tilde{s} + \tilde{P}\tilde{y} \in S$  since  $\bar{s}_{(p^*, 1)} \in \mathbb{Z}_+$  and  $R\bar{s} + P\bar{y} \in S \times \mathbb{Z}_+$ . Thus  $(\tilde{s}, \tilde{y}) \in X_S(\tilde{R}, \tilde{P})$ . A direct calculation shows

$$\begin{aligned} & \sum_{r \in \mathcal{R}} \hat{\psi}((r, 0)) \bar{s}_r + \sum_{p \in \mathcal{P}} \hat{\pi}(p) \bar{y}_p \\ &= \sum_{r \in \mathcal{R} \setminus \{(p^*, 1)\}} \hat{\psi}((r, 0)) \bar{s}_r + \hat{\psi}((p^*, 1)) \bar{s}_{(p^*, 1)} + \sum_{p \in \mathcal{P}} \hat{\pi}(p) \bar{y}_p \\ &= \sum_{r \in \mathcal{R} \setminus \{(p^*, 1)\}} \hat{\psi}((r, 0)) \bar{s}_r + V_\psi(p^*) \bar{s}_{(p^*, 1)} + \sum_{p \in \mathcal{P}} \hat{\pi}(p) \bar{y}_p \\ &= \sum_{r \in \tilde{R}} \psi(r) \tilde{s}_r + \sum_{p \in \tilde{P}} \pi(p) \tilde{y}_p \\ &\geq 1, \end{aligned}$$

where the inequality comes from the validity of  $(\psi, \pi)$ .  $\square$

Theorem 5 states there exist functions  $\psi' : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$  and  $\pi' : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$  such that  $(\psi', \pi')$  is a minimal valid pair for  $(S \times \mathbb{Z}_+), \mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}_+$  whose restriction to  $\mathbb{R}^n$  dominate  $(\psi, \pi)$  (because the restriction dominates  $(\hat{\psi}, \hat{\pi})$ ). Since  $\psi$  is a minimal valid function for  $S$ , the restriction of  $\psi'$  to  $\mathbb{R}^n$  must match  $\psi$ . Similarly, since  $\pi$  is a minimal lifting of  $\psi$ ,  $\pi'$  restricted to  $\mathbb{R}^n$  must coincide with  $\pi$ . This also implies that

$$(4.4) \quad \psi'((p^*, 1)) = \hat{\psi}((p^*, 1)) = V_\psi(p^*)$$

from the construction of  $B(V_\psi(p^*), p^*)$ . Now  $\psi'((r, 0)) \leq \psi_{B(V_\psi(p^*), p^*)}(r)$  for  $r \in \mathbb{R}^n$ .

Indeed, by Lemma 11 there exists a  $r' \in \mathbb{R}^n$  and  $\mu \geq 0$  such that  $(r, 0) = (r', 0) + \mu(p^*, 1)$  and  $\psi_{B(V_\psi(p^*), p^*)}((r', 0)) + \mu\psi_{B(V_\psi(p^*), p^*)}((p^*, 1)) = \psi_{B(V_\psi(p^*), p^*)}((r, 0))$ . A direct calculation gives

$$\begin{aligned}
 \psi'((r, 0)) &\leq \psi'((r', 0)) + \mu\psi'((p^*, 1)) && \text{by Theorem 3} \\
 &= \hat{\psi}((r', 0)) + \mu\hat{\psi}((p^*, 1)) && \text{by Equation (4.4)} \\
 (4.5) \quad &= \psi_{B(V_\psi(p^*), p^*)}((r', 0)) + \mu\psi_{B(V_\psi(p^*), p^*)}((p^*, 1)) \\
 &= \psi_{B(V_\psi(p^*), p^*)}((r, 0)).
 \end{aligned}$$

Now let  $p \in \mathbb{R}^n$ . Since  $\pi'$  dominates  $\pi$  when restricted to  $\mathbb{R}^n$  it follows that  $\pi(p) \leq \pi'((p, 0))$ . Using this followed by Proposition 2 then Equation 4.5, we see that

$$\begin{aligned}
 \pi(p) &\leq \pi'((p, 0)) \\
 &\leq \inf\{\psi'((p, 0) + (w, z)) : (w, z) \in W_{S \times \mathbb{Z}_+}^+, (p, 0) + (w, z) \in \mathbb{R}^n \times \mathbb{R}_+\} \\
 &\leq \inf\{\psi_{B(V_\psi(p^*), p^*)}((p, 0) + (w, z)) : (w, z) \in W_{S \times \mathbb{Z}_+}^+, (p, 0) + (w, z) \in \mathbb{R}^n \times \mathbb{R}_+\} \\
 &= \psi_{B(V_\psi(p^*), p^*)}^*((p, 0)).
 \end{aligned}$$

This is the desired result.  $\square$

**4.2. Towards a description of the fixing region.** In this section, we will start with a maximal  $S$ -free convex 0-neighborhood  $B$  and a point  $p^*$ . We then define a collection of polyhedra (given by explicit inequalities) whose union will be shown to be a subset of  $\mathcal{F}_{\psi, p^*}$ , where  $\psi$  is defined from  $B$  using (4.1).

Let  $\tilde{B} = \{r \in \mathbb{R}^d : a^i \cdot r \leq 1 \ i \in I\}$  be a polyhedral 0-neighborhood<sup>2</sup>. For  $x \in \mathbb{R}^d$ , the *spindle corresponding to  $x$*  is defined as

$$(4.6) \quad R_{\tilde{B}}(x) := \{r \in \mathbb{R}^d : (a^i - a^k) \cdot r \leq 0, (a^i - a^k) \cdot (x - r) \leq 0, \forall i \in I\},$$

where  $\psi(x) = a^k \cdot x$ . The original motivation for this definition was the following observation made in [2, 17]:

**OBSERVATION 14.** *Let  $\psi$  be a valid function for  $S$  obtained from a maximal  $S$ -free 0-neighborhood  $B$  as in (4.1), and let  $\bar{x} \in B \cap S$ . If  $p^* \in R_B(\bar{x})$ , then  $V_\psi(p^*) = \psi(p^*)$  and  $(\bar{x}, 1)$  is a blocking point for  $B(V_\psi(p^*), p^*)$ .*

**DEFINITION 15.** *Let  $(\bar{x}, \bar{x}_{n+1}) \in (S \times \mathbb{Z}_+) \cap B(V_\psi(p^*), p^*)$  be a blocking point for  $B(V_\psi(p^*), p^*)$ . The set  $R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1}) \subseteq \mathbb{R}^n \times \mathbb{R}$  is the  $n + 1$ -dimensional spindle corresponding to  $(\bar{x}, \bar{x}_{n+1})$ .*

For  $t \in \mathbb{R}$  define  $H_t := \{(x, t) \in \mathbb{R}^{n+1}\}$ . The following proposition, whose proof appears in the Appendix, states that translating  $H_0 \cap R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1})$  by  $tp^*$  is equivalent to projecting the ‘height- $t$  slice’  $H_t \cap R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1})$  onto the first  $n$ -coordinates.

**PROPOSITION 16.** *Let  $(\bar{x}, \bar{x}_{n+1}) \in (S \times \mathbb{Z}_+) \cap B(V_\psi(p^*), p^*)$  be a blocking point and  $t \in \mathbb{R}$ . Then*

$$H_t \cap R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1}) = (H_0 \cap R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1})) + t(p^*, 1).$$

<sup>2</sup>We intentionally use  $d$  for the dimension and  $\tilde{B}$  instead of  $B$ , because the set  $R_{\tilde{B}}(x)$  will be applied to sets in  $d = n + 1$  and  $d = n + 2$  dimensions which are derived from a set  $B$  in  $\mathbb{R}^n$  using equation (4.2).

Our geometric inner approximation of  $\mathcal{F}_{\psi, p^*}$  is the content of the next theorem.

**THEOREM 17.** *Let  $\psi$  be a valid function for  $S$  obtained from a maximal  $S$ -free 0-neighborhood  $B$  as in (4.1). Let  $p^* \in \mathbb{R}^n$  and  $(w, N)$  be a maximizer in (3.1). Then*

$$(4.7) \quad (R_B(w) \cup (R_B(w) + p^*) \cup \dots \cup (R_B(w) + Np^*)) + W_S \subseteq \mathcal{F}_{\psi, p^*}.$$

*Equivalently, let  $(\bar{x}, \bar{x}_{n+1})$  be a blocking point for  $B(V_\psi(p^*), p^*)$ . Then (4.7) holds for  $(w, N) = (\bar{x} - \bar{x}_{n+1}p^*, \bar{x}_{n+1})$ .*

*Moreover, for any  $\pi \in \mathcal{L}_{\psi, p^*}$ ,  $p \in R_B(w) \cup (R_B(w) + p^*) \cup \dots \cup (R_B(w) + Np^*)$ , and  $w \in W_S$ , we have  $\pi(p + w) = \pi(p) = \psi_{B(V_\psi(p^*), p^*)}^*((p, 0))$ , where  $\psi_{B(V_\psi(p^*), p^*)}^*$  is the function defined in (4.3).*

An important result in previous literature [17, 16, 13, 2] is that for every  $\bar{x} \in S \cap B$ ,  $R_B(\bar{x}) + W_S \subseteq R(B)$ , where  $R(B)$  is the extended lifting region as in (1.3). Of course, for every  $p^* \in \mathbb{R}^n$ ,  $R(B) \subseteq \mathcal{F}_{\psi, p^*}$ . Thus we obtain the following corollary of Theorem 17.

**COROLLARY 18.** *Let  $\psi$  be a valid function for  $S$  obtained from a maximal  $S$ -free 0-neighborhood  $B$  as in (4.1). For any  $(\bar{x}, \bar{x}_{n+1}) \in (S \times \mathbb{Z}_+) \cap B(V_\psi(p^*), p^*)$ , let  $(w, N) = (\bar{x} - \bar{x}_{n+1}p^*, \bar{x}_{n+1})$ . Then*

$$(4.8) \quad (R_B(w) \cup (R_B(w) + p^*) \cup \dots \cup (R_B(w) + Np^*)) + W_S \subseteq \mathcal{F}_{\psi, p^*}.$$

Note that  $(\bar{x}, \bar{x}_{n+1})$  in the above Corollary need not be a blocking point as  $\bar{x}_{n+1}$  could be 0.

The remainder of this section is dedicated to the proof of Theorem 17. Before proving the result, we build a few tools. For  $p_1^*, p_2^* \in \mathbb{R}^n$ , consider sequentially lifting  $p_1^*$  then  $p_2^*$ . Define

$$(4.9) \quad V_\psi(p_2^*; p_1^*) := \inf \{ \pi(p_2^*) : \pi \in \mathcal{L}_{\psi, p_1^*} \}.$$

The geometric construction used for  $V_\psi(p_1^*)$  may be extended to calculate  $V_\psi(p_2^*; p_1^*)$ . Intuitively,  $V_\psi(p_1^*)$  is found by constructing a translated cone in  $\mathbb{R}^{n+1}$  with base  $B$ , and  $V_\psi(p_2^*; p_1^*)$  is found similarly by constructing a translated cone in  $\mathbb{R}^{n+2}$  with base  $B(V_\psi(p_1^*), p_1^*)$ . For  $\lambda > 0$ , define

$$(4.10) \quad \begin{aligned} B(\lambda, p_2^*; V_\psi(p_1^*)) &:= \\ \{ (r, r_{n+1}, r_{n+2}) \in \mathbb{R}^{n+2} : a^i \cdot r + (V_\psi(p_1^*) - a^i \cdot p_1^*)r_{n+1} + (\lambda - a^i \cdot p_2^*)r_{n+2} \leq 1 \}. \end{aligned}$$

The following result confirms the geometric approach is valid for finding  $V_\psi(p_2^*; p_1^*)$ .

**PROPOSITION 19.**

$$(4.11) \quad V_\psi(p_2^*; p_1^*) = \inf \{ \lambda > 0 : B(\lambda, p_2^*; V_\psi(p_1^*)) \text{ is } S \times \mathbb{Z}_+ \times \mathbb{Z}_+ \text{ free} \}.$$

The proof of Proposition 19 is similar to the reasoning in [13] that leads to Proposition 7 and is therefore relegated to Appendix A.2.

A consequence of the following proposition is that  $V_\psi(p_2^*; p_1^*)$  is invariant under certain translations of  $p_2^*$ .

**PROPOSITION 20.** *Let  $S \subseteq \mathbb{R}^n \setminus \{0\}$  be a closed subset and let  $\hat{B} \subseteq \mathbb{R}^{n+1}$  be a maximal  $S \times \mathbb{Z}_+$ -free convex 0-neighborhood. Let  $\hat{\psi}$  be the corresponding function derived using (4.1). Consider any  $\hat{p} \in \mathbb{R}^{n+1}$  and let  $(\bar{x}, \bar{x}_{n+1}, 1) \in S \times \mathbb{Z}_+ \times \mathbb{Z}_+$  be a blocking point of  $\hat{B}(V_{\hat{\psi}}(\hat{p}), \hat{p})$ . Let  $w \in \mathbb{Z}$  such that  $(\bar{x}, \bar{x}_{n+1} + w, 1) \in S \times \mathbb{Z}_+ \times \mathbb{Z}_+$ . Then  $V_{\hat{\psi}}(\hat{p} + (0_n, w)) = V_{\hat{\psi}}(\hat{p})$ , where  $0_n$  is the zero vector in  $\mathbb{R}^n$ .*

*Proof.* By Proposition 7, it is sufficient to show

$$\begin{aligned} & \inf \left\{ \lambda > 0 : \hat{B}_{\hat{\psi}}(\lambda, \hat{p}) \text{ is } S \times \mathbb{Z}_+ \times \mathbb{Z}_+ \text{ free} \right\} \\ &= \inf \left\{ \lambda > 0 : \hat{B}_{\hat{\psi}}(\lambda, \hat{p} + (0_n, w)) \text{ is } S \times \mathbb{Z}_+ \times \mathbb{Z}_+ \text{ free} \right\}. \end{aligned}$$

Observe that the sets  $\hat{B}_{\hat{\psi}}(\lambda, \hat{p})$  and  $\hat{B}_{\hat{\psi}}(\lambda, \hat{p} + (0_n, w))$  are translated cones in  $\mathbb{R}^{n+2}$ . The geometric interpretation of the equality above is that the ratio of the ‘lifting’ vector to the apex is preserved between the two cones. The idea of the proof is to create a unimodular transformation between the two cones that preserves this ratio.

Define the linear transformation  $U : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$  by

$$U(y, y_{n+1}, y_{n+2}) = (y, y_{n+1} + y_{n+2}w, y_{n+2}),$$

where  $y \in \mathbb{R}^n$  and  $y_{n+1}, y_{n+2} \in \mathbb{R}$ . Note that  $U$  is invertible and  $U^{-1}(y, y_{n+1}, y_{n+2}) = (y, y_{n+1} - y_{n+2}w, y_{n+2})$ . Furthermore,  $w \in \mathbb{Z}$  by definition, and so  $U$  is unimodular. In the following arguments, it is useful to note  $U(\hat{p}, 1) = (\hat{p} + (0_n, w), 1)$ .

Since  $U$  is unimodular and is the identity map when restricted to  $\mathbb{R}^n \times \{0\} \times \{0\}$ , it maps  $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$  free sets to  $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$  free sets. Therefore, since  $(\hat{p}, 1)$  and  $(\hat{p} + (0_n, w), 1)$  define  $\hat{B}_{\hat{\psi}}(\lambda, \hat{p})$  and  $\hat{B}_{\hat{\psi}}(\lambda, \hat{p} + (0_n, w))$ , respectively, we have

$$U\left(\hat{B}_{\hat{\psi}}(\lambda, \hat{p})\right) \subseteq \hat{B}_{\hat{\psi}}(V_{\hat{\psi}}(\hat{p} + (0_n, w)), \hat{p} + (0_n, w))$$

for each value of  $\lambda$ . Similarly, as  $U^{-1}$  is unimodular, it follows that

$$U^{-1}\left(\hat{B}_{\hat{\psi}}(\lambda, \hat{p} + (0_n, w))\right) \subseteq \hat{B}_{\hat{\psi}}(V_{\hat{\psi}}(\hat{p}), \hat{p})$$

for each  $\lambda$ . Thus  $U\left(\hat{B}_{\hat{\psi}}(V_{\hat{\psi}}(\hat{p}), \hat{p})\right) = \hat{B}_{\hat{\psi}}(V_{\hat{\psi}}(\hat{p} + (0_n, w)), \hat{p} + (0_n, w))$ . Since  $U$  is unimodular, ratios of vector magnitudes are preserved and the result follows.  $\square$

*Proof of Theorem 17.* The equivalence of the two statements is a consequence of Lemma 9: if  $(w, N)$  is a maximizer in (3.1), then  $(w + Np^*, N)$  is a blocking point for  $B(V_{\psi}(p^*), p^*)$ , and conversely, if  $(\bar{x}, \bar{x}_{n+1})$  is a blocking point for  $B(V_{\psi}(p^*), p^*)$ , then  $(w, N) = (\bar{x} - \bar{x}_{n+1}p^*, \bar{x}_{n+1})$  is a maximizer in (3.1). Thus, it suffices to prove the result for a blocking point  $(\bar{x}, \bar{x}_{n+1})$  and  $(w, N) = (\bar{x} - \bar{x}_{n+1}p^*, \bar{x}_{n+1})$ .

To reduce notational baggage, we introduce  $\hat{B} := B(V_{\psi}(p^*), p^*)$  and let  $\hat{\psi} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  denote the function obtained by applying (4.1) with  $B = \hat{B}$ . Recall also the function  $\psi_{B(V_{\psi}(p^*), p^*)}^*$  defined in (4.3) derived from this function  $\hat{\psi}$ . Consider any  $q \in R_B(w) \cup (p^* + R_B(w)) \cup \dots (Np^* + R_B(w))$  and any  $\pi \in \mathcal{L}_{\psi, p^*}$ . By definition,  $V_{\psi}(q; p^*) \leq \pi(q)$ , independent of  $\pi$ . Therefore it is sufficient to show that this inequality holds at equality.

Let  $q \in jp^* + R_B(w)$  for some  $j \in \{0, 1, \dots, N\}$ . Observe that

$$V_{\psi}(q; p^*) \leq \pi(q) \leq \psi_{B(V_{\psi}(p^*), p^*)}^*\left(\binom{q}{0}\right) \leq \inf_{n \in \mathbb{Z}_+} \hat{\psi}\left(\binom{q}{n}\right) \leq \hat{\psi}\left(\binom{q}{j}\right),$$

where the second inequality follows from Theorem 12. By the definition of  $w$  and Proposition 16,  $(q, j) \in R_{\hat{B}}(\bar{x}, \bar{x}_{n+1}) \cap H_j$ . Thus Observation 14 implies

$$\hat{\psi}\left(\binom{q}{j}\right) = V_{\hat{\psi}}\left(\binom{q}{j}\right),$$

and  $(\bar{x}, \bar{x}_{n+1}, 1)$  is a blocking point for  $\hat{B}\left(V_{\hat{\psi}}\left(\begin{smallmatrix} q \\ j \end{smallmatrix}\right), \begin{smallmatrix} q \\ j \end{smallmatrix}\right)\right)$ . Since  $N - j \geq 0$  and  $(\hat{x}, N) + (0_d, -j) \in S \times \mathbb{Z}_+$ , we can apply Propositions 19 and 20 to conclude that

$$V_{\hat{\psi}}\left(\begin{smallmatrix} q \\ j \end{smallmatrix}\right) = V_{\hat{\psi}}\left(\begin{smallmatrix} q \\ 0 \end{smallmatrix}\right) = V_{\psi}(q; p^*).$$

Combining the inequalities and equalities,  $V_{\psi}(q; p^*) = \pi(q) = \psi_{B(V_{\psi}(p^*), p^*)}^*((q, 0))$ .  $\square$

**4.3. Translation invariance of fixing region.** Fix a maximal  $S$ -free convex 0-neighborhood  $B$ . From Corollary 18, it is clear that if  $p^*$  is chosen such that

$$(4.12) \quad \left( \bigcup_{(\bar{x}, \bar{x}_{n+1}) \in B(V_{\psi}(p^*), p^*) \cap (S \times \mathbb{Z}_+)} \left( \bigcup_{i=0}^{\bar{x}_{n+1}} (R_B(\bar{x} - \bar{x}_{n+1}p^*) + ip^*) \right) \right) + W_S = \mathbb{R}^n,$$

then  $\mathcal{L}_{\psi, p^*}$  is a singleton. In other words, after fixing the coefficient for  $p^*$ , all other lifting coefficients are fixed. Let us introduce a more compact notation:

$$(4.13) \quad \mathcal{X}(B, p^*) := \left( \bigcup_{(\bar{x}, \bar{x}_{n+1}) \in B(V_{\psi}(p^*), p^*) \cap (S \times \mathbb{Z}_+)} \left( \bigcup_{i=0}^{\bar{x}_{n+1}} (R_B(\bar{x} - \bar{x}_{n+1}p^*) + ip^*) \right) \right)$$

**THEOREM 21.** *Let  $B$  be a maximal  $S$ -free convex 0-neighborhood and let  $m \in \mathbb{R}^n$  such that  $0 \in \text{int}(B + m)$ ; thus  $B + m$  is a maximal  $S + m$ -free convex 0-neighborhood. For  $p^* \in \mathbb{R}^n$  and  $\hat{p} := p^* + V_{\psi}(p^*)m \in \mathbb{R}^n$ ,*

$$\mathcal{X}(B, p^*) + W_S = \mathbb{R}^n$$

*if and only if*

$$\mathcal{X}(B + m, \hat{p}) + W_{S+m} = \mathbb{R}^n.$$

In other words, if for a given maximal  $S$ -free convex 0-neighborhood  $B$ , there exists a  $p^*$  that makes  $B$  one point fixable, then for any translation  $B + m$ , there exists a  $\hat{p}$  that makes  $B + m$  one point fixable.

The proof of Theorem 21 is very technical in nature and is similar to that of Theorem 3.1 in [9]. For this reason, we relegate the proof to Appendix A.3.

**5. Application: Fixing Regions of Type 3 triangles.** In this section, we will utilize the technology developed in the previous sections to discuss minimal liftings for Type 3 triangles in  $\mathbb{R}^2$  (which are defined precisely below). The first result (Proposition 22) shows that for any Type 3 triangle, there exists a point  $p^*$  such that after lifting  $p^*$  to its minimal value, an entire ball around  $p^*$  is fixed, i.e., there is a ball around  $p^*$  that is a subset of  $\mathcal{F}_{\psi, p^*}$ , where  $\psi$  is the minimal valid function coming from the Type 3 triangle. This is analogous to one of the main results from [13] concerning the extended lifting region. We next identify sufficient conditions that guarantee that a given Type 3 triangle is one point fixable (Proposition 27). Finally, we show that a family of Type 3 triangles that arise in the extensively studied mixing set problem satisfies this sufficient condition, and therefore all triangles in this family are one point fixable. This forms the content of Section 5.2.

In this section, we let  $S = \mathbb{Z}^2 + b$  for  $b = (b_1, b_2) \notin \mathbb{Z}^2$ . Without loss of generality, we can assume  $-1 \leq b_1, b_2 \leq 0$ . Moreover, by relabeling the coordinates, we can further assume without loss of generality  $-1 \leq b_2 \leq b_1 \leq 0$ . This means that the

origin  $(0, 0)$  is contained in the triangle  $\text{conv}\{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$ , where  $\bar{s}_1 = (1+b_1, 1+b_2)$ ,  $\bar{s}_2 = (b_1, 1+b_2)$ , and  $\bar{s}_3 = (b_1, b_2)$ .

Let  $\gamma_1, \gamma_2, \gamma_3 > 0$  with  $\gamma_2, \gamma_3 < 1$ . Define the vectors

$$(5.1) \quad \omega^1 = \omega^1(\gamma_1) := \left( \frac{1}{(1, \gamma_1) \cdot (b_1 + 1, b_2 + 1)}, \frac{\gamma_1}{(1, \gamma_1) \cdot (b_1 + 1, b_2 + 1)} \right),$$

$$(5.2) \quad \omega^2 = \omega^2(\gamma_2) := \left( \frac{-1}{(-1, \gamma_2) \cdot (b_1, b_2 + 1)}, \frac{\gamma_2}{(-1, \gamma_2) \cdot (b_1, b_2 + 1)} \right),$$

$$(5.3) \quad \omega^3 = \omega^3(\gamma_3) := \left( \frac{\gamma_3}{(\gamma_3, -1) \cdot (b_1, b_2)}, \frac{-1}{(\gamma_3, -1) \cdot (b_1, b_2)} \right),$$

and the triangle  $T(\gamma_1, \gamma_2, \gamma_3) \subseteq \mathbb{R}^2$  by

$$(5.4) \quad T(\gamma_1, \gamma_2, \gamma_3) := \left\{ (x_1, x_2) \in \mathbb{R}^2 : \omega_1^i x_1 + \omega_2^i x_2 \leq 1, i \in \{1, 2, 3\} \right\}.$$

The family of triangles  $T(\gamma_1, \gamma_2, \gamma_3)$  with  $\gamma_1, \gamma_2, \gamma_3 > 0$  and  $\gamma_2, \gamma_3 < 1$  are all maximal  $S$ -free convex 0-neighborhoods, and the three sides contain the points  $\bar{s}_1, \bar{s}_2$  and  $\bar{s}_3$  from  $S$  respectively in their relative interiors. In the literature, they are referred to as the Type 3 family of maximal  $S$ -free triangles.

*Notation.* We use  $\text{relint}(X)$  and  $\text{rec}(X)$  to denote the relative interior and recession cone of any closed, convex body  $X$ , respectively. For any  $v \in \mathbb{R}^d$ ,  $v\mathbb{R}$  will denote the line  $\{\lambda v : \lambda \in \mathbb{R}\}$  and  $v\mathbb{R}_+$  will denote the ray  $\{\lambda v : \lambda \in \mathbb{R}_+\}$ . For  $\varepsilon > 0$  and  $x \in \mathbb{R}^d$ , let  $D(x; \varepsilon) = \{y \in \mathbb{R}^d : \|x - y\| < \varepsilon\}$ .

### 5.1. Fixing Regions of general Type 3 triangles.

**5.1.1. Nonempty interior for the fixing region.** The next result says that for any Type 3 triangle, there always exists a  $p^* \in \mathbb{R}^2$  that fixes a set of measure greater than 0.

**PROPOSITION 22.** *Let  $T$  be a Type 3 triangle as described above and let  $\psi$  be the valid function derived from  $T$  using (4.1). There exists  $p^* \in \mathbb{R}^2$  and an  $\varepsilon > 0$  such  $D(p^*; \varepsilon) \subseteq \mathcal{F}_{\psi, p^*}$ .*

Proposition 22 is a simple consequence of the following.

**PROPOSITION 23.** *Let  $T \subseteq \mathbb{R}^2$  be as above. There exists  $P \subseteq \mathbb{R}^3$  such that*

- (i)  $P$  is a translated cone with three facets and an apex  $a = (a_1, a_2, a_3)$  satisfying  $a_3 > 0$ ,
- (ii)  $P \cap \{x \in \mathbb{R}^3 : x_3 = 0\} = T \times \{0\}$ ,
- (iii)  $P$  is maximal  $S \times \mathbb{Z}_+$  free, and
- (iv) each facet of  $P$  contains a point  $(s_i, z_i) \in S \times \mathbb{Z}_+$ ,  $i = 1, 2, 3$ , with  $z_i \geq 1$ , in its relative interior.

Proposition 22 follows from Proposition 23 by setting  $p^* \in \mathbb{R}^2$  such that the ray spanned by  $(p^*, 1)$  passes through the apex of  $P$ . Then one observes that  $(R_T(s_1 - z_1 p^*) + p^*) \cup (R_T(s_2 - z_2 p^*) + p^*) \cup (R_T(s_3 - z_3 p^*) + p^*)$  contains  $p^*$  in its interior, and  $(R_T(s_1 - z_1 p^*) + p^*) \cup (R_T(s_2 - z_2 p^*) + p^*) \cup (R_T(s_3 - z_3 p^*) + p^*) \subseteq \mathcal{F}_{\psi, p^*}$  by Theorem 17 and the fact that  $z_i \geq 1$ .

The idea for constructing  $P$  in Proposition 23 is to first extend the three edges of  $T$  to hyperplanes in  $\mathbb{R}^3$  and create a translated cone satisfying (i), (ii), and (iii).

We then ‘rotate’ the hyperplanes one at a time until (iv) is satisfied. This rotation preserves (i), (ii), and (iii).

Recall  $\omega^i$  in (5.1)-(5.3). For  $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$  and  $\beta > \frac{1+2\gamma_1+\gamma_2-\gamma_2\gamma_3-\gamma_1\gamma_2\gamma_3}{\gamma_1+\gamma_2} > 0$ , define the vectors

$$\begin{aligned} n_{\alpha_1} &= \left( \omega_1^1, \omega_2^1, \frac{-\gamma_1}{(1-\alpha_1)(1, \gamma_1) \cdot (b_1+1, b_2+1)} \right), \\ n_{\alpha_2} &= \left( \omega_1^2, \omega_2^2, \frac{\alpha_2}{(\alpha_2-1)(-1, \gamma_2) \cdot (b_1, b_2+1)} \right), \\ n_{\alpha_3} &= \left( \omega_1^3, \omega_2^3, \frac{\beta(1-\alpha_3)}{(\gamma_3, -1) \cdot (b_1, b_2)} \right), \end{aligned}$$

the hyperplanes

$$\begin{aligned} H_1(\alpha_1) &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : n_{\alpha_1} \cdot (x_1, x_2, x_3) \leq 1\}, \\ H_2(\alpha_2) &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : n_{\alpha_2} \cdot (x_1, x_2, x_3) \leq 1\}, \\ H_3(\alpha_3) &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : n_{\alpha_3} \cdot (x_1, x_2, x_3) \leq 1\}, \end{aligned}$$

and the polyhedron

$$P(\alpha_1, \alpha_2, \alpha_3) := H_1(\alpha_1) \cap H_2(\alpha_2) \cap H_3(\alpha_3).$$

The definition of  $\beta$  implies that  $P(0, 0, 0)$  is a translated cone with apex  $(a_1, a_2, a_3)$ , and  $a_3 \in (0, 1)$ . Whenever  $P(\alpha_1, \alpha_2, \alpha_3)$  is a translated cone, we will use  $(a_1, a_2, a_3) \in \mathbb{R}^3$  to denote the apex and  $F_1(\alpha_1, \alpha_2, \alpha_3)$ ,  $F_2(\alpha_1, \alpha_2, \alpha_3)$ , and  $F_3(\alpha_1, \alpha_2, \alpha_3)$  to denote the facets defined by  $H_1(\alpha_1)$ ,  $H_2(\alpha_2)$  and  $H_3(\alpha_3)$ , respectively.

Let  $\alpha, \alpha^* \in [0, 1]$  be such that  $\alpha \leq \alpha^*$ ,  $\alpha_2, \alpha_3 \in [0, 1]$ , and set  $H_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \geq 0\}$ . In this situation, observe that

$$\text{OBSERVATION 24. } P(\alpha, \alpha_2, \alpha_3) \cap H_+ \subseteq P(\alpha^*, \alpha_2, \alpha_3) \cap H_+.$$

Observation 24 is about  $P(\alpha_1, \alpha_2, \alpha_3)$  when  $\alpha_1$  is allowed to vary in  $[0, 1]$  that follows from direct computation. Similar statements can be made when  $\alpha_2$  or  $\alpha_3$  is allowed to vary instead of  $\alpha_1$ .

Here are two more properties about  $P(\alpha_1, \alpha_2, \alpha_3)$  for  $\alpha_3 \in [0, 1]$ .

CLAIM 25. Suppose  $P(\alpha_1, \alpha_2, \alpha_3)$  is  $S \times \mathbb{Z}_+$  free. If  $(p_1, p_2, 1) \in \text{rec}(P(\alpha_1, \alpha_2, \alpha_3))$  then  $(p_1, p_2, 1) \in \mathbb{Z}^3$ .

*Proof of Claim 25.* Assume to the contrary that  $(p_1, p_2, 1) \in \mathbb{R}^3 \setminus \mathbb{Z}^3$ . Since  $(b_1, b_2, 0), (b_1, b_2 + 1, 0), (b_1 + 1, b_2 + 1, 0) \in P(\alpha_1, \alpha_2, \alpha_3)$ , either  $(b_1 + p_1 - \lfloor p_1 \rfloor, b_2 + p_2 - \lfloor p_2 \rfloor, 0) \in \text{int}(P(\alpha_1, \alpha_2, \alpha_3))$  or  $(b_1 + 1, b_2 + 1, 0) - (p_1 - \lfloor p_1 \rfloor, p_2 - \lfloor p_2 \rfloor, 0) \in \text{int}(P(\alpha_1, \alpha_2, \alpha_3))$ . Suppose that  $(b_1 + p_1 - \lfloor p_1 \rfloor, b_2 + p_2 - \lfloor p_2 \rfloor, 0) \in \text{int}(P(\alpha_1, \alpha_2, \alpha_3))$  (a similar argument can be made in the other case). Therefore, there exists an open ball  $B \subseteq \mathbb{R}^3$  centered at  $(b_1 + p_1 - \lfloor p_1 \rfloor, b_2 + p_2 - \lfloor p_2 \rfloor, 0)$  and contained in  $\text{int}(P(\alpha_1, \alpha_2, \alpha_3))$ .

Consider the cylinder  $C := B + (p_1, p_2, 1)\mathbb{R}$ . Note that  $B + (p_1, p_2, 1)\mathbb{R}_+ \subseteq \text{int}(P(\alpha_1, \alpha_2, \alpha_3))$ ,  $C$  is symmetric about  $(b_1 - \lfloor p_1 \rfloor, b_2 - \lfloor p_2 \rfloor, -1)$ , and  $\text{vol}(C) = \infty$ . Therefore, by Minkowski’s Convex Body Theorem, there exists a point  $(z_1, z_2, z_3) \in (S \times \mathbb{Z}) \cap C$  with  $z_3 \geq 0$ . However, this implies that  $(z_1, z_2, z_3) \in (S \times \mathbb{Z}_+) \cap \text{int}(P(\alpha_1, \alpha_2, \alpha_3))$ , contradicting that  $P(\alpha_1, \alpha_2, \alpha_3)$  is  $S \times \mathbb{Z}_+$  free.  $\square$



CLAIM 26. Assume that  $P(0, 0, \alpha_3)$  is  $S \times \mathbb{Z}_+$  free for  $\alpha_3 \in [0, 1)$ . Then  $\alpha_3 \leq 1 - \frac{1-\gamma_3}{\beta}$ . Furthermore, if there also exists  $(z_1, z_2, 1) \in (S \times \mathbb{Z}_+) \cap F_3(0, 0, \alpha_3)$  then equality holds and  $(z_1, z_2, 1) = (b_1 + 1, b_2 + 1, 1)$ .

*Proof of Claim 26.* If  $\alpha_3 > 1 - \frac{1-\gamma_3}{\beta}$  then  $(b_1 + 1, b_2 + 1, 1) \in \text{int}(P(0, 0, \alpha_3))$ , contradicting that  $P(0, 0, \alpha_3)$  is  $S \times \mathbb{Z}_+$  free (this can be seen since  $n_{\alpha_i} \cdot (b_1 + 1, b_2 + 1, 1) < 1$  for each  $i$ ).

Now suppose that there exists  $(z_1, z_2, 1) \in (S \times \mathbb{Z}_+) \cap \text{relint}(F_3(0, 0, \alpha_3))$ . Suppose that  $\alpha_3 = 1 - \frac{1-\gamma_3}{\beta}$ . A direct calculation shows that  $(b_1 + 1, b_2 + 1, 1)$  is contained in  $\text{relint}(F_3)$  and

$$F_3(0, 0, \alpha_3) \cap \{(x_1, x_2, x_3) : x_3 = 1\} \subseteq C_1 \cup C_2,$$

where

$$\begin{aligned} C_1 &:= \{(x_1, x_2, 1) : -b_1 + b_2 \leq -x_1 + x_2 \leq 1 - b_1 + b_2\} \\ C_2 &:= \{(x_1, x_2, 1) : 1 + b_2 \leq x_2 \leq 2 + b_2\}. \end{aligned}$$

Furthermore, it can be seen that  $(F_3(0, 0, \alpha_3) \cap \{(x_1, x_2, x_3) : x_3 = 1\}) \setminus \{(b_1 + 1, b_2 + 1, 1)\}$  is contained in  $\text{relint}(C_1) \cup \text{relint}(C_2)$  and so  $(b_1 + 1, b_2 + 1, 1)$  is the only  $S \times \mathbb{Z}_+$  point in  $(F_3(0, 0, \alpha_3) \cap \{(x_1, x_2, x_3) : x_3 = 1\})$ . Hence the result holds when equality holds.

If  $\alpha_3 < 1 - \frac{1-\gamma_3}{\beta}$  then

$$\begin{aligned} &\text{relint}(F_3(0, 0, \alpha_3) \cap \{(x_1, x_2, x_3) : x_3 = 1\}) \\ &\subseteq \text{relint}(P_3(0, 0, 1 - \frac{1-\gamma_3}{\beta}) \cap \{(x_1, x_2, x_3) : x_3 = 1\}). \end{aligned}$$

Hence  $\text{relint}(F_3)$  contains no  $\mathbb{Z}^2 \times \mathbb{Z}_+$  points, which is a contradiction.  $\square$

*Proof of Proposition 23.* Let  $P_\emptyset := P(0, 0, 0)$ . Observe that  $P_\emptyset$  satisfies (i), (ii), and (iii). Indeed, by the choice of  $\beta$ ,  $P_\emptyset$  is a translated cone such that (i) holds. Furthermore, the definitions of  $P_\emptyset$  and  $T$  imply that  $P_\emptyset$  has  $T$  as its base. Hence (ii) holds. Finally, in order to see that (iii) holds, let  $(\bar{s}, \bar{z}) \in (S \times \mathbb{Z}_+) \cap P_\emptyset$ . By the choice of  $\beta$ , we have  $a_3 \in (0, 1)$  and so  $\bar{z} = 0$ . However, since  $P_\emptyset$  has  $T$  as its base and  $T$  is a maximal lattice-free triangle,  $\bar{s}$  must be contained in the boundary of  $T$ , and therefore  $(\bar{s}, \bar{z}) \notin \text{int}(P_\emptyset)$ . Hence  $P_\emptyset$  is  $S \times \mathbb{Z}_+$ -free. The maximality of  $S \times \mathbb{Z}_+$  free sets follows from the characterization provided in [2].

Let  $P_{\{3\}} := P(0, 0, \alpha_3^*)$ , where  $\alpha_3^*$  is defined by

$$\alpha_3^* := \sup \{\alpha \in [0, 1) : P(0, 0, \alpha) \text{ is } S \times \mathbb{Z}_+ \text{ free}\}.$$

We claim that  $P_{\{3\}}$  satisfies (i), (ii), (iii), and (iv) for  $F_3(0, 0, \alpha_3^*)$ . By definition,  $P_{\{3\}}$  satisfies (ii) and (iii). Suppose that  $P_{\{3\}}$  satisfies (i). Then  $F_3(0, 0, \alpha_3^*) \cap \{(x_1, x_2, x_3) : x_3 \geq 0\}$  is compact and there exists a point  $(s, z) \in S \times \mathbb{Z}_+$  that is closest to  $F_3(0, 0, \alpha_3^*)$ . Therefore, by the definition of  $\alpha_3^*$ ,  $P_{\{3\}}$  satisfies (iv) for  $F_3(0, 0, \alpha_3^*)$ . Hence, in order to prove the claim, it is sufficient to show  $P_{\{3\}}$  satisfies (i). Assume to the contrary that there exists  $(p_1, p_2, 1) \in \text{rec}(P_{\{3\}})$ . From Claims 25 and 26, we have that  $(p_1, p_2, 1) = (1, 1, 1)$ . However, this implies that  $(b_1, b_2 + 1, 0) + (p_1, p_2, 1) = (b_1 + 1, b_2 + 2, 1)$  is in  $F_2(0, 0, \alpha_3^*)$ , and, through a direct calculation, it can be seen that  $(b_1 + 1, b_2 + 2, 1) \notin F_2(0, 0, \alpha_3^*)$ . This is a contradiction, and so  $P_{\{3\}}$  satisfies (i).

Suppose that  $(r_1, r_2, r_3)$  is a blocking point on  $F_3(0, 0, \alpha_3^*)$  arising in the construction of  $P_{\{3\}}$ . Define  $P_{\{2,3\}} := P(0, \alpha_2^*, \alpha_3^*)$ , where  $\alpha_2^*$  is defined by

$$\alpha_2^* := \sup \{ \alpha \in [0, 1] : P(0, \alpha, \alpha_3^*) \text{ is } S \times \mathbb{Z}_+ \text{ free} \}.$$

We claim that  $P_{\{2,3\}}$  satisfies (i), (ii), (iii), and (iv) for  $F_2(0, \alpha_2^*, \alpha_3^*)$  and  $F_3(0, \alpha_2^*, \alpha_3^*)$ . By definition,  $P_{\{2,3\}}$  satisfies (ii) and (iii), and similar to above, if  $P_{\{2,3\}}$  satisfies (i) then it satisfies (iv) for  $F_2(0, \alpha_2^*, \alpha_3^*)$ . Finally, from Observation 24,  $(r_1, r_2, r_3) \in \text{relint}(F_3(0, \alpha_2^*, \alpha_3^*))$ . Hence, it is sufficient to show that  $P_{\{2,3\}}$  satisfies (i).

Suppose to the contrary that  $P_{\{2,3\}}$  does not satisfy (i). Then, from Claims 25 and 26,  $\text{rec}(P_{\{2,3\}})$  equals the cone generated by some  $(p_1, p_2, 1) \in \mathbb{Z}^3$ . From (5.4) and Observation 24,

$$(b_1, b_2, 0) \in \text{relint}(F_3(0, 0, \alpha_3^*)) \subseteq \text{relint}(F_3(0, \alpha_2^*, \alpha_3^*)).$$

Moreover, assumption (ii) indicates that  $(b_1, b_2, 0)$  is the only point in  $(S \times \{0\}) \cap \text{relint}(F_3(0, \alpha_2^*, \alpha_3^*))$ . Therefore, since  $\text{rec}(P_{\{2,3\}})$  is generated by  $(p_1, p_2, 1)$ , there exists exactly one point in  $(S \times \{k\}) \cap \text{relint}(F_3(0, \alpha_2^*, \alpha_3^*))$  for each  $k \in \mathbb{Z}_+$ , and such a point is of the form  $(b_1, b_2, 0) + k(p_1, p_2, 1)$ . In particular,

$$(5.5) \quad (r_1, r_2, r_3) = (b_1, b_2, 0) + r_3(p_1, p_2, 1).$$

Since  $(r_1, r_2, r_3)$  is a blocking point,  $r_3 \geq 1$ . However, as  $(r_1, r_2, r_3), (b_1, b_2, 0) \in \text{relint}(F_3(0, 0, \alpha_3^*))$ , (5.5) implies that  $(b_1, b_2, 0) + k(p_1, p_2, 1) \in \text{relint}(F_3(0, 0, \alpha_3^*))$  for  $1 \leq k \leq r_3$ . In particular,  $(b_1, b_2, 0) + (p_1, p_2, 1) \in \text{relint}(F_3(0, 0, \alpha_3^*))$ . Using Claim 26, we see  $(p_1, p_2, 1) = (1, 1, 1)$ , and again using (5.4),  $(b_1, b_2 + 1, 0) \in \text{relint}(F_2(0, \alpha_2^*, \alpha_3^*))$ . Therefore

$$\begin{aligned} (b_1, b_2 + 1, 0) + (p_1, p_2, 1) &= (b_1, b_2 + 1, 0) + (1, 1, 1) \\ &= (b_1 + 1, b_2 + 2, 1) \\ &\in \text{relint}(F_2(0, \alpha_2^*, \alpha_3^*)). \end{aligned}$$

However, a direct calculation shows that this is not the case. This is a contradiction and so  $P_{\{2,3\}}$  satisfies (i).

The final facet to tilt is  $F_1(0, \alpha_2^*, \alpha_3^*)$ . Define  $P_{\{1,2,3\}} := P(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ , where  $\alpha_1^*$  is defined by

$$\alpha_1^* := \sup \{ \alpha \in [0, 1] : P(\alpha, \alpha_2^*, \alpha_3^*) \text{ is } S \times \mathbb{Z}_+ \text{ free} \}.$$

Using the same argument as for  $P_{\{2,3\}}$ , we have that  $P_{\{1,2,3\}}$  satisfies (i), (ii), (iii). Furthermore,  $P_{\{1,2,3\}}$  satisfies (iv) for  $F_1(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ ,  $F_2(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ , and  $F_3(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ .  $\square$

### 5.1.2. Sufficient condition for Type 3 triangles to be one point fixable.

Let  $T := T(\gamma_1, \gamma_2, \gamma_3)$  be a Type 3 triangle given by (5.4) with appropriate values for  $\gamma_1, \gamma_2, \gamma_3$ . Let  $P \subseteq \mathbb{R}^3$  be the polyhedron defined by

$$(5.6) \quad P := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \begin{aligned} &\frac{1}{(1, \gamma_1) \cdot (b_1 + 1, b_2 + 1)} x_1 + \frac{\gamma_1}{(1, \gamma_1) \cdot (b_1 + 1, b_2 + 1)} x_2 + \left( 1 - \frac{(1, \gamma_1) \cdot (b_1 + 1, b_2 + 2)}{(1, \gamma_1) \cdot (b_1 + 1, b_2 + 1)} \right) x_3 \leq 1, \\ &-\frac{1}{(-1, \gamma_2) \cdot (b_1, b_2 + 1)} x_1 + \frac{\gamma_2}{(-1, \gamma_2) \cdot (b_1, b_2 + 1)} x_2 \leq 1, \\ &\frac{\gamma_3}{(\gamma_3, -1) \cdot (b_1, b_2)} x_1 - \frac{1}{(\gamma_3, -1) \cdot (b_1, b_2)} x_2 + \left( \frac{1}{2} - \frac{(\gamma_3, -1) \cdot (1 + b_1, 2 + b_2)}{2(\gamma_3, -1) \cdot (b_1, b_2)} \right) x_3 \leq 1 \end{aligned} \right\}.$$

Note that  $T \times \{0\} = P \cap \{(x_1, x_2, x_3) : x_3 = 0\}$ , and  $P$  contains the  $S \times \mathbb{Z}_+$  points  $(s_1, z_1) := (1 + b_1, 1 + b_2, 0)$ ,  $(s_2, z_2) := (b_1, 1 + b_2, 0)$ ,  $(s_3, z_3) := (b_1, b_2, 0)$ ,  $(s_4, z_4) := (1 + b_1, 2 + b_2, 1)$ ,  $(s_5, z_5) := (b_1, 1 + b_2, 1)$ , and  $(s_6, z_6) := (1 + b_1, 1 + b_2, 2)$ . Furthermore  $P$  has three facets,  $F_1, F_2$ , and  $F_3$ , containing the points  $\{(s_1, z_1), (s_4, z_4)\}$ ,  $\{(s_2, z_2), (s_5, z_5)\}$ , and  $\{(s_3, z_3), (s_6, z_6)\}$ , respectively.

In the situation of Type 3 triangles,  $S = \mathbb{Z}^2 + b$  and  $W_S = \mathbb{Z}^2$ . Assuming a certain relation of  $\gamma_1, \gamma_2, \gamma_3$ ,  $P$  is a translated cone with apex contained in  $\mathbb{R}^2 \times \mathbb{R}_+$ . If  $P$  is also  $S \times \mathbb{Z}_+$  free then it is possible to find a  $p^*$  such that  $\mathcal{X}(T, p^*) + \mathbb{Z}^2 = \mathbb{R}^2$  (recall Equation (4.13)). This implies that  $\mathcal{L}_{\psi, p^*}$  is a singleton, and thus  $T$  is one point fixable. This is the content of Proposition 27.

**PROPOSITION 27.** *Let  $T$  and  $P$  be described as above with  $\gamma_1, \gamma_2, \gamma_3 > 0$  and  $\gamma_2, \gamma_3 < 1$ . Let  $\psi$  be the valid function for  $S$  obtained from  $T$  using (4.1). Then the following hold.*

- (i)  *$P$  is a translated cone whose apex  $a^* = (a_1^*, a_2^*, a_3^*)$  satisfies  $a_3^* > 0$  if and only if  $\gamma_2(2 - \gamma_3 + 2\gamma_1\gamma_3) - \gamma_1\gamma_3 > 0$ .*
- (ii) *If  $P$  is  $S \times \mathbb{Z}_+$ -free then setting  $p^* = \frac{1}{a_3^*}(a_1^*, a_2^*)$  implies  $\mathcal{X}(T, p^*) + W_S = \mathbb{R}^n$  and consequently,  $\mathcal{L}_{\psi, p^*}$  consists of a unique lifting function.*

*Proof.* We can symbolically compute  $a^* = (a_1^*, a_2^*, a_3^*) = F_1 \cap F_2 \cap F_3$ :

$$a^* := \begin{pmatrix} b_1 + \frac{\gamma_2(2+2\gamma_1-\gamma_3)}{\gamma_2(2-\gamma_3+2\gamma_1\gamma_3)-\gamma_1\gamma_3}, \\ b_2 + \frac{\gamma_1(2-\gamma_3+2\gamma_2\gamma_3)-(1+\gamma_2)(-2+\gamma_3)}{\gamma_2(2-\gamma_3+2\gamma_1\gamma_3)-\gamma_1\gamma_3}, \\ \frac{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)}{\gamma_2(2-\gamma_3+2\gamma_1\gamma_3)-\gamma_1\gamma_3} \end{pmatrix}.$$

In order for  $P$  to be translated cone with apex in the upper-half space, it is equivalent to show that  $2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3) > 0$  and  $\gamma_2(2 - \gamma_3 + 2\gamma_1\gamma_3) - \gamma_1\gamma_3 > 0$ . The first inequality holds since  $\gamma_3 < 1$  while the second holds by hypothesis. Hence (i) holds.

By Proposition 6,  $\mathcal{L}_{\psi, p^*}$  is nonempty. According to Theorem 17, in order to see that  $\mathcal{L}_{\psi, p^*}$  is indeed a unique lifting function, it is sufficient to show that  $\mathcal{X}(T, p^*) + \mathbb{Z}^2 = \mathbb{R}^2$  (recall  $W_S = \mathbb{Z}^2$ ). We draw inspiration from [17]. The crucial observation is that for the choice of  $p^*$  in the hypothesis,  $P = B(V_\psi(p^*), p^*)$ .

Figure 8 in [17] labels the vertices of the spindles  $R_T(s_1)$ ,  $R_T(s_2)$  and  $R_T(s_3)$  for  $T$ . For completeness, we reproduce the labels in Figure 1; the values  $v_i$  and  $\delta_i$  are defined below.

The vertices of  $T$  are

$$\begin{aligned} v_1 &= \left( b_1 + \frac{1 + \gamma_1}{1 + \gamma_1\gamma_3}, b_2 + \frac{\gamma_3 + \gamma_1\gamma_3}{1 + \gamma_1\gamma_3} \right) \\ v_2 &= \left( b_1 + \frac{\gamma_2}{\gamma_1 + \gamma_2}, b_2 + \frac{1 + \gamma_1 + \gamma_2}{\gamma_1 + \gamma_2} \right) \\ v_3 &= \left( b_1 + \frac{-\gamma_2}{1 - \gamma_2\gamma_3}, b_2 + \frac{-\gamma_2\gamma_3}{1 - \gamma_2\gamma_3} \right), \end{aligned}$$

and the values of  $\delta_i$  for  $i \in \{1, 2, 3\}$  in Figure 1 are

$$\delta_1 = \frac{1 + \gamma_1\gamma_3}{1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3}, \quad \delta_2 = \frac{\gamma_1 + \gamma_2}{1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3}, \quad \delta_3 = \frac{1 + \gamma_1 - \gamma_2\gamma_3 - \gamma_1\gamma_2\gamma_3}{1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3}.$$

The  $\delta_i$ 's are convex coefficients so that  $s_i = \delta_i v_i + (1 - \delta_i) v_{i+1}$  for  $i = 1, 2, 3$ , where  $v_4$

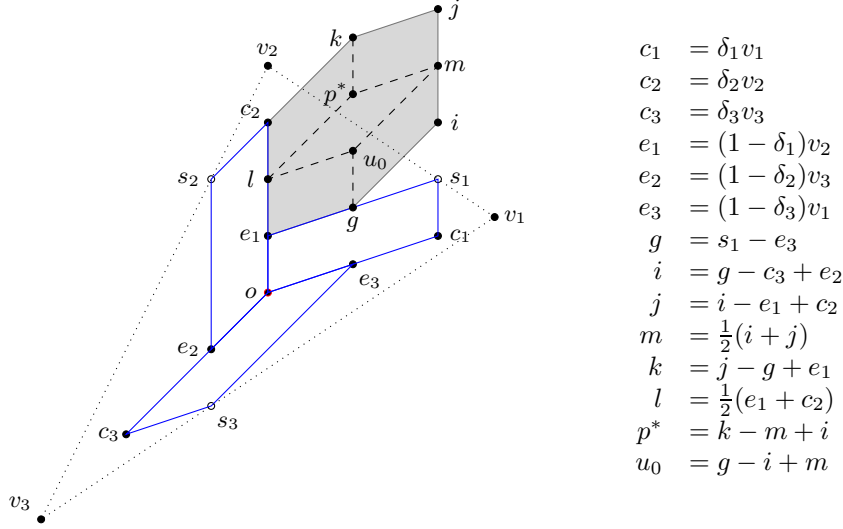


FIG. 1. The spindles of  $T$  [17]. The region  $K := \text{conv}\{c_2, k, j, i, g, e_1\}$  is shaded, and  $o$  denotes the origin. The vertices  $v_i$  and the convex coefficients  $\delta_i$  are defined below.

is interpreted as  $v_1$ . One observes that  $\delta_i \in [0, 1]$  holds because  $\gamma_i > 0$  and  $\gamma_2, \gamma_3 < 1$ .

Define the region  $K := \text{conv}\{c_2, k, j, i, g, e_1\}$  (see Figure 1). The literature [17, 13, 2] contains results that show  $\mathbb{R}^2 \setminus (K + \mathbb{Z}^2) \subseteq \mathcal{X}(T, p^*) + \mathbb{Z}^2$ . Hence, to complete the proof it suffices to show that  $K \subseteq \mathcal{X}(T, p^*) + \mathbb{Z}^2$ , because then  $K + \mathbb{Z}^2 \subseteq \mathcal{X}(T, p^*) + \mathbb{Z}^2$ . For this, write  $K$  as  $K = \cup_{i=1}^5 K_i$ , where

$$\begin{aligned}
K_1 &= \text{conv}\{l, e_1, g, u_0\} \\
K_2 &= \text{conv}\{u_0, m, i, g\} \\
K_3 &= \text{conv}\{m, j, k, v_0\} \\
K_4 &= \text{conv}\{c_2, k, v_0, l\} \\
K_5 &= \text{conv}\{l, v_0, m, u_0\}.
\end{aligned}$$

From Theorem 17,  $R_T(s_4 - p^*)$ ,  $R_T(s_4 - p^*) + p^*$ ,  $R_T(s_5 - p^*) + (1, 1)$ ,  $R_T(s_5 - p^*) + p^*$ , and  $R_T(s_6 - 2p^*) + p^*$  are contained in  $\mathcal{X}(T, p^*) + \mathbb{Z}^2$ . The following claim completes the proof.

CLAIM 28.  $K_1 \subseteq R_T(s_4 - p^*)$ ,  $K_2 \subseteq R_T(s_5 - p^*) + (1, 1)$ ,  $K_3 \subseteq R_T(s_4 - p^*) + p^*$ ,  $K_4 \subseteq R_T(s_5 - p^*) + p^*$ , and  $K_5 \subseteq R_T(s_6 - 2p^*) + p^*$ .

The proof of Claim 28 appears in Appendix B.  $\square$

**5.2. Type 3 triangles from the mixing set.** Proposition 27 assumes that the pyramid  $P$  is  $S \times \mathbb{Z}_+$  free. This is the situation for Type 3 triangles derived from the *mixing set* [16, 20]. The mixing set Type 3 triangles are defined for  $S = b + \mathbb{Z}^2$  where  $b \in \text{int}(\text{conv}((0, -1), (0, -1/2), (-1, -1)))$ , which is a subset of our earlier restriction  $-1 < b_2 < b_1 < 0$ , with the additional constraint that  $b_1 - 2b_2 > 1$ . Define  $\delta_b = -b_1^2 - b_2^2 + b_1 b_2 - b_2$ . Observe  $\delta_b = b_1(b_2 - b_1) - b_2(1 + b_2) > 0$ .

Consider the set  $T(b) \subseteq \mathbb{R}^2$  defined by

$$T(b) := \left\{ (x_1, x_2) \in \mathbb{R}^2 : \begin{aligned} &\left(\frac{-b_1}{\delta_b}\right) x_1 + \left(\frac{b_1 - b_2}{\delta_b}\right) x_2 \leq 1, \\ &\left(\frac{-b_1 - 1}{\delta_b}\right) x_1 + \left(\frac{b_1 - b_2}{\delta_b}\right) x_2 \leq 1, \\ &\left(\frac{-b_1}{\delta_b}\right) x_1 + \left(\frac{b_1 - b_2 - 1}{\delta_b}\right) x_2 \leq 1 \end{aligned} \right\}.$$

It can be checked directly that  $T(b)$  is a Type 3 triangle by setting  $\gamma_1 = \frac{b_2 - b_1}{b_1}$ ,  $\gamma_2 = \frac{b_1 - b_2}{1 + b_1}$ ,  $\gamma_3 = \frac{b_1}{b_1 - b_2 - 1}$ . Note that the constraints on  $b$  imply that  $\gamma_1, \gamma_2, \gamma_3 > 0$  and  $\gamma_2, \gamma_3 < 1$ , as required. By construction,  $T(b) \cap S = \{(b_1, b_2), (b_1, 1 + b_2), (1 + b_1, 1 + b_2)\}$ . Plugging in these values of  $\gamma_1, \gamma_2, \gamma_3$  in (5.6) we obtain

$$(5.7) \quad \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \begin{aligned} &\left(\frac{-b_1}{\delta_b}\right) x_1 + \left(\frac{b_1 - b_2}{\delta_b}\right) x_2 - \left(\frac{b_1 - b_2}{\delta_b}\right) x_3 \leq 1, \\ &\left(\frac{-b_1 - 1}{\delta_b}\right) x_1 + \left(\frac{b_1 - b_2}{\delta_b}\right) x_2 \leq 1, \\ &\left(\frac{-b_1}{\delta_b}\right) x_1 + \left(\frac{b_1 - b_2 - 1}{\delta_b}\right) x_2 + \left(\frac{2 - b_1 + 2b_2}{2\delta_b}\right) x_3 \leq 1 \end{aligned} \right\}.$$

Let  $P(b)$  denote the set in Equation 5.7.

We verify the two conditions in Proposition 27, concluding that there always exists a  $p^* \in \mathbb{R}^2$  satisfying one point fixability for mixing set triangles. The condition  $\gamma_2(2 - \gamma_3 + 2\gamma_1\gamma_3) - \gamma_1\gamma_3 > 0$  can be checked by using the values  $\gamma_1 = \frac{b_2 - b_1}{b_1}$ ,  $\gamma_2 = \frac{b_1 - b_2}{1 + b_1}$ ,  $\gamma_3 = \frac{b_1}{b_1 - b_2 - 1}$ , and the constraints  $-1 < b_2 < b_1 < 0$ . We verify that  $\text{int}(P(b)) \cap (S \times \mathbb{Z}_+) = \emptyset$  in the next proposition.

**PROPOSITION 29.** *For all  $b \in \text{int}(\text{conv}((0, 0), (0, -1/2), (-1, -1)))$  it follows that  $\text{int}(P(b)) \cap (S \times \mathbb{Z}_+) = \emptyset$ .*

*Proof.* For  $k \in \mathbb{Z}_+$ , let  $H_k := \{(x_1, x_2, k) \in \mathbb{R}^3\}$ . Recall that  $P(b) \cap H_0 = T(b) \times \{0\}$ , which is an  $S$ -free triangle. Thus we only need to show  $\text{relint}(P(b) \cap H_k) \cap (S \times \{k\}) = \emptyset$  for  $k \geq 1$ .

For a fixed  $k \geq 1$ , define the split sets

$$\begin{aligned} C_1 &:= \{(x_1, x_2, k) \in \mathbb{R}^3 : k \leq x_2 \leq k + 1\} + (b_1, b_2, 0) \\ C_2 &:= \{(x_1, x_2, k) \in \mathbb{R}^3 : 0 \leq -2x_1 + x_2 \leq 1\} + (b_1, b_2, 0) \\ C_3 &:= \left\{ (x_1, x_2, k) \in \mathbb{R}^3 : \frac{k}{2} \leq -x_1 + x_2 \leq \frac{k}{2} + \frac{1}{2} \right\} + (b_1, b_2, 0). \end{aligned}$$

Note that for each  $k \geq 1$ , the splits  $C_1, C_2$  and  $C_3$  have no  $S \times \{k\}$  points in their relative interior. Hence if we can show that  $\text{relint}(P(b) \cap H_k) \subseteq \text{relint}(C_1) \cup \text{relint}(C_2) \cup \text{relint}(C_3)$ , then we will be done; to this end, suppose  $(x_1^*, x_2^*, k) \in \text{relint}(P(b) \cap H_k)$  but not in  $\text{relint}(C_1) \cup \text{relint}(C_2)$ . This implies that  $(x_1^*, x_2^*, k)$  does not strictly satisfy one of the inequalities defining  $C_1$  and one of the inequalities defining  $C_2$ . This leads to four cases.

*Case 1* Suppose  $x_2^* - b_2 \leq k$  and  $-2(x_1^* - b_1) + (x_2^* - b_2) \leq 0$ . Observe that

$$\begin{aligned}
& \left( \frac{-b_1}{\delta_b} \right) x_1^* + \left( \frac{b_1 - b_2 - 1}{\delta_b} \right) x_2^* + \left( \frac{2 - b_1 + 2b_2}{2\delta_b} \right) \\
& \geq \left( \frac{-b_1}{\delta_b} \right) \left( \frac{2b_1 + x_2^* - b_2}{2} \right) + \left( \frac{b_1 - b_2 - 1}{\delta_b} \right) x_2^* + \left( \frac{2 - b_1 + 2b_2}{2\delta_b} \right) \\
& = \left( \frac{b_1 - 2b_2 - 2}{2\delta_b} \right) x_2^* + \left( \frac{2 - b_1 + 2b_2}{2\delta_b} \right) k + \left( \frac{-2b_2^2 + b_1 b_2}{2\delta_b} \right) \\
& \geq \left( \frac{b_1 - 2b_2 - 2}{2\delta_b} \right) (k + b_2) + \left( \frac{2 - b_1 + 2b_2}{2\delta_b} \right) k + \left( \frac{-2b_2^2 + b_1 b_2}{2\delta_b} \right) \\
& = 1.
\end{aligned}$$

The first inequality follows from the assumption  $-2(x_1^* - b_1) + (x_2^* - b_2) \leq 0$ , and the second inequality follows from the assumption  $x_2^* - b_2 \leq k$ . This contradicts that  $(x_1^*, x_2^*, k) \in \text{relint}(P(b) \cap H_k)$  because the third inequality defining  $P(b)$  is violated.

*Case 2* Suppose  $x_2^* - b_2 \leq k$  and  $-2(x_1^* - b_1) + (x_2^* - b_2) \geq 1$ . We claim that  $(x_1^*, x_2^*, k) \in \text{relint}(C_3)$ . For this, it is sufficient to show that  $\frac{k}{2} < -(x_1^* - b_1) + (x_2^* - b_2) < \frac{k}{2} + \frac{1}{2}$ . Note that since  $(x_1^*, x_2^*, k) \in \text{relint}(P(b) \cap H_k)$ , the third inequality defining  $P(b)$  gives the following bound on  $x_2^*$

$$x_2^* > \frac{-b_1}{1 + b_2 - b_1} x_1^* + \frac{k}{2} + \frac{1 + b_2}{2(1 + b_2 - b_1)} k + \frac{-\delta_b}{1 + b_2 - b_1}.$$

Using this, we see that

$$\begin{aligned}
& -(x_1^* - b_1) + (x_2^* - b_2) \\
& > -(x_1^* - b_1) + \left( \frac{-b_1}{1 + b_2 - b_1} x_1^* + \frac{k}{2} + \frac{1 + b_2}{2(1 + b_2 - b_1)} k + \frac{-\delta_b}{1 + b_2 - b_1} \right) - b_2 \\
& = \frac{k}{2} + \left( \frac{-1 - b_2}{1 + b_2 - b_1} \right) x_1^* + \left( \frac{1 + b_2}{2(1 + b_2 - b_1)} \right) k + \left( \frac{b_1 + b_1 b_2}{1 + b_2 - b_1} \right) \\
& \geq \frac{k}{2} + \left( \frac{-1 - b_2}{1 + b_2 - b_1} \right) \left( \frac{x_2^* - b_2 - 1 + 2b_1}{2} \right) + \left( \frac{1 + b_2}{2(1 + b_2 - b_1)} \right) k + \left( \frac{b_1 + b_1 b_2}{1 + b_2 - b_1} \right) \\
& = \frac{k}{2} + \left( \frac{-1 - b_2}{2(1 + b_2 - b_1)} \right) x_2^* + \left( \frac{1 + b_2}{2(1 + b_2 - b_1)} \right) k + \left( \frac{2b_2 + b_2^2 + 1}{2(1 + b_2 - b_1)} \right) \\
& \geq \frac{k}{2} + \left( \frac{-1 - b_2}{2(1 + b_2 - b_1)} \right) (k + b_2) + \left( \frac{1 + b_2}{2(1 + b_2 - b_1)} \right) k + \left( \frac{2b_2 + b_2^2 + 1}{2(1 + b_2 - b_1)} \right) \\
& = \frac{k}{2} + \frac{1 + b_2}{2(1 + b_2 - b_1)} \\
& > \frac{k}{2}.
\end{aligned}$$

The second inequality follows since  $-2(x_1^* - b_1) + (x_2^* - b_2) \geq 1$  and  $\frac{-1 - b_2}{-b_1 + b_2 + 1} < 0$ , the third inequality follows since  $x_2^* \leq k + b_2$ , and the fourth inequality follows since  $\frac{1 + b_2}{2(1 + b_2 - b_1)} > 0$ .

Since  $(x_1^*, x_2^*, k) \in \text{relint}(P(b) \cap H_k)$ , the second inequality defining  $P(b)$  implies

$$\begin{aligned}
-(x_1^* - b_1) + (x_2^* - b_2) &< -x_1^* + b_1 + \left( \frac{\delta_b}{b_1 - b_2} + \frac{1+b_1}{b_1 - b_2} x_1^* \right) - b_2 \\
&= \left( \frac{1+b_2}{b_1 - b_2} \right) x_1^* + \frac{-b_2 - b_1 b_2}{b_1 - b_2} \\
&\leq \left( \frac{1+b_2}{b_1 - b_2} \right) \left( \frac{2b_1 + x_2^* - b_2 - 1}{2} \right) + \frac{-b_2 - b_1 b_2}{b_1 - b_2} \\
&= \left( \frac{1+b_2}{2(b_1 - b_2)} \right) x_2^* + \left( \frac{2b_1 - 4b_2 - b_2^2 - 1}{2(b_1 - b_2)} \right) \\
&\leq \left( \frac{1+b_2}{2(b_1 - b_2)} \right) (k + b_2) + \left( \frac{2b_1 - 4b_2 - b_2^2 - 1}{2(b_1 - b_2)} \right) \\
&= \frac{k}{2} + \left( \frac{1 - b_1 + 2b_2}{b_1 - b_2} \right) \frac{k}{2} + \left( \frac{2b_1 - 3b_2 - 1}{2(b_1 - b_2)} \right) \\
&\leq \frac{k}{2} + \left( \frac{1 - b_1 + 2b_2}{b_1 - b_2} \right) \frac{1}{2} + \left( \frac{2b_1 - 3b_2 - 1}{2(b_1 - b_2)} \right) \\
&= \frac{k}{2} + \frac{1}{2}.
\end{aligned}$$

The second inequality follows since  $-2(x_1^* - b_1) + (x_2^* - b_2) \geq 1$  and  $\frac{1+b_2}{b_1+b_2+1} > 0$ , the third inequality follows since  $x_2^* \leq k + b_2$ , and the fourth inequality follows since  $k \geq 1$  and  $1 < b_1 - 2b_2$ .

*Case 3* Suppose  $x_2^* - b_2 \geq k + 1$  and  $-2(x_1^* - b_1) + (x_2^* - b_2) \leq 0$ . Observe

$$\begin{aligned}
&\left( \frac{-b_1}{\delta_b} \right) x_1^* + \left( \frac{b_1 - b_2}{\delta_b} \right) x_2^* - \left( \frac{b_1 - b_2}{\delta_b} \right) k \\
&\geq \left( \frac{-b_1}{\delta_b} \right) \left( \frac{2b_1 + x_2^* - b_2}{2} \right) + \left( \frac{b_1 - b_2}{\delta_b} \right) x_2^* - \left( \frac{b_1 - b_2}{\delta_b} \right) k \\
&= \left( \frac{b_1 - 2b_2}{2\delta_b} \right) x_2^* - \left( \frac{b_1 - b_2}{\delta_b} \right) k + \left( \frac{-2b_1^2 + b_1 b_2}{2\delta_b} \right) \\
&\geq \left( \frac{b_1 - 2b_2}{2\delta_b} \right) (k + 1 + b_2) - \left( \frac{b_1 - b_2}{\delta_b} \right) k + \left( \frac{-2b_1^2 + b_1 b_2}{2\delta_b} \right) \\
&= \left( \frac{-b_1}{2\delta_b} \right) k + \left( \frac{b_1}{2\delta_b} \right) + 1 \\
&\geq \left( \frac{-b_1}{2\delta_b} \right) + \left( \frac{b_1}{2\delta_b} \right) + 1 \\
&= 1.
\end{aligned}$$

The first inequality follows since  $\frac{-b_1}{\delta_b} > 0$  and  $-2(x_1^* - b_1) + (x_2^* - b_2) \geq 0$ , the second inequality follows since  $b_1 - 2b_2 > 1$  and  $x_2^* \geq k + 1 + b_2$ , and the third inequality follows since  $k \geq 1$ . This contradicts that  $(x_1^*, x_2^*, k) \in \text{relint}(P(b) \cap H_k)$  because the first inequality defining  $P(b)$  is violated.

*Case 4* Suppose  $x_2^* - b_2 \geq k + 1$  and  $-2(x_1^* - b_1) + (x_2^* - b_2) \geq 1$ . Observe that

$$\begin{aligned}
\left( \frac{-b_1 - 1}{\delta_b} \right) x_1^* + \left( \frac{b_1 - b_2}{\delta_b} \right) x_2^* &\geq \left( \frac{-b_1 - 1}{\delta_b} \right) \left( \frac{x_2^* - 1 + 2b_1 - b_2}{2} \right) + \left( \frac{b_1 - b_2}{\delta_b} \right) x_2^* \\
&= \left( \frac{b_1 - 2b_2 - 1}{2\delta_b} \right) x_2^* + \left( \frac{-b_1 - 1}{\delta_b} \right) \left( \frac{2b_1 - b_2 - 1}{2} \right) \\
&\geq \left( \frac{b_1 - 2b_2 - 1}{2\delta_b} \right) (2 + b_2) + \left( \frac{-b_1 - 1}{\delta_b} \right) \left( \frac{2b_1 - b_2 - 1}{2} \right) \\
&= 1 + \frac{b_1 - 2b_2 - 1}{2\delta_b} \\
&> 1.
\end{aligned}$$

The first inequality comes from  $\frac{-b_1 - 1}{\delta_b} < 0$  and  $-2(x_1^* - b_1) + (x_2^* - b_2) \geq 1$ . The second inequality comes from the fact that  $b_1 - 2b_2 > 1$  and  $\delta_b > 0$  so the term  $\left( \frac{b_1 - 2b_2 - 1}{2\delta_b} \right)$  is positive; furthermore,  $x_2^* \geq k + 1 + b_2 \geq 2 + b_2 > 0$  since  $k \geq 1$  and  $-1 < b_2$ . The last inequality follows since  $\delta_b > 0$  and  $b_1 - 2b_2 > 1$ . This contradicts that  $(x_1^*, x_2^*, k) \in \text{relint}(P(b) \cap H_k)$  because the second inequality defining  $P(b)$  is violated.  $\square$

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## Appendix A. Appendix.

### A.1. Proof of Proposition 16.

*Proof of Proposition 16.* It follows from (4.2) and (4.6) that

$$\begin{aligned} & R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1}) \\ = & \left\{ (r, r_{n+1}) : \begin{aligned} & (a^i - a^k) \cdot r + r_{n+1}((a^k - a^i) \cdot p^*) \leq 0 \\ & (a^i - a^k) \cdot (\bar{x} - r) + (\bar{x}_{n+1} - r_{n+1})((a^k - a^i) \cdot p^*) \leq 0 \end{aligned} \quad \forall i \in I \right\}, \end{aligned}$$

where  $k \in I$  is such that  $\psi(x) = a^k \cdot x$ . Therefore,

$$\begin{aligned} & H_t \cap R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1}) \\ = & \left\{ (r, t) : \begin{aligned} & (a^i - a^k) \cdot r + t((a^k - a^i) \cdot p^*) \leq 0 \\ & (a^i - a^k) \cdot (\bar{x} - r) + (\bar{x}_{n+1} - t)((a^k - a^i) \cdot p^*) \leq 0 \end{aligned} \quad \forall i \in I \right\} \\ = & \left\{ (r, t) : \begin{aligned} & (a^i - a^k) \cdot (r - tp^*) \leq 0 \\ & (a^i - a^k) \cdot (\bar{x} - (r - tp^*)) + \bar{x}_{n+1}((a^k - a^i) \cdot p^*) \leq 0 \end{aligned} \quad \forall i \in I \right\} \\ = & \left\{ (\tilde{r} + tp^*, t) : \begin{aligned} & (a^i - a^k) \cdot \tilde{r} \leq 0 \\ & (a^i - a^k) \cdot (\bar{x} - \tilde{r}) + \bar{x}_{n+1}((a^k - a^i) \cdot p^*) \leq 0 \end{aligned} \quad \forall i \in I \right\} \\ = & (H_0 \cap R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1})) + t(p^*, 1). \end{aligned}$$

□

### A.2. Proof of Proposition 19.

*Proof of Proposition 19.* Observe that if  $B(\lambda, p_2^*; V_\psi(p_1^*))$  is  $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$  free for some  $\lambda > 0$  then it is also maximal  $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$  free. This follows from the characterization of maximal S-free sets given in [3].

Consider the model

$$(A.1) \quad \left\{ (s, y_1, y_2) \in \mathbb{R}_+^n \times \mathbb{Z}_+ \times \mathbb{Z}_+ : \sum_{r \in \mathbb{R}^n} r s_r + p_1^* y_{p_1^*} + p_2^* y_{p_2^*} \in S \right\}$$

and note that  $(s, y_1, y_2) \in (A.1)$  if and only if  $(s, y_1, y_2)$  is in

$$(A.2) \quad \left\{ (s, y_1, y_2) \in \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+ : \sum_{r \in \mathbb{R}^n} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} s_r + \begin{pmatrix} p_1^* \\ 1 \\ 0 \end{pmatrix} y_{p_1^*} + \begin{pmatrix} p_2^* \\ 0 \\ 1 \end{pmatrix} y_{p_2^*} \in S \times \mathbb{Z}_+ \times \mathbb{Z}_+ \right\}.$$

CLAIM 30. Let  $\lambda > 0$ . If the inequality

$$(A.3) \quad \sum_{r \in \mathbb{R}^n} \psi(r) s_r + V_\psi(p_1^*) y_{p_1^*} + \lambda y_{p_2^*} \geq 1$$

is valid for (A.1) then  $B(\lambda, p_2^*; V_\psi(p_1^*))$  is  $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$  free.

*Proof of Claim 30.* Take  $(\bar{x}, \bar{x}_{n+1}, \bar{x}_{n+2}) \in S \times \mathbb{Z}_+ \times \mathbb{Z}_+$ . Let  $\bar{r} = \bar{x} - \bar{x}_{n+1} p_1^* + \bar{x}_{n+2} p_2^*$ ,  $\bar{z}_1 = \bar{x}_{n+1}$ ,  $\bar{z}_2 = \bar{x}_{n+2}$  and  $\bar{s}_r = 1$  if  $r = \bar{r}$  and  $\bar{s}_r = 0$  otherwise. Note that

$$\sum_{r \in \mathbb{R}^n} r \bar{s}_r + p_1^* \bar{z}_1 + p_2^* \bar{z}_2 = \bar{x} \in S.$$

Since (A.3) is valid for (A.1), it follows that

$$\begin{aligned}
1 &\leq \sum_{r \in \mathbb{R}^n} \psi(r) \bar{s}_r + V_\psi(p_1^*) \bar{z}_1 + \lambda \bar{z}_2 \\
&= \psi(\bar{r}) + V_\psi(p_1^*) \bar{x}_{n+1} + \lambda \bar{x}_{n+2} \\
&= \max_i \{a_i \cdot (\bar{x} - \bar{x}_{n+1} p_1^* - \bar{x}_{n+2} p_2^*) + V_\psi(p_1^*) \bar{x}_{n+1} + \lambda \bar{x}_{n+2}\} \\
&= \max_i \{a_i \cdot \bar{x} + (V_\psi(p_1^*) - a_i \cdot p_1^*) \bar{x}_{n+1} + (\lambda - a_i \cdot p_2^*) \bar{x}_{n+2}\}.
\end{aligned}$$

Hence  $B$  is  $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$  free.  $\square$

The converse of the Claim 30 is also true.

CLAIM 31. Let  $\lambda > 0$ . If  $B(\lambda, p_2^*; V_\psi(p_1^*))$  is  $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$  free then (A.3) is valid for (A.1).

*Proof of Claim 31.* Consider the function

$$\Psi(r, r_{n+1}, r_{n+2}) = \max_{i \in I} \{a^i \cdot r + (V_\psi(p_1^*) - a^i \cdot p_1^*) r_{n+1} + (\lambda - a^i \cdot p_2^*) r_{n+2}\}.$$

Take  $(s, y_1, y_2)$  satisfying (A.1). From the observation above,  $(s, y_1, y_2)$  also satisfies (A.2). Note that  $\Psi(r, 0, 0) = \psi(r)$ ,  $\Psi(p_1^*, 1, 0) = V_\psi(p_1^*)$ , and  $\Psi(p_2^*, 0, 1) = \lambda$ . It follows that

$$\begin{aligned}
&\sum_{r \in \mathbb{R}^n} \psi(r) s_r + V_\psi(p_1^*) y_1 + \lambda y_2 \\
&= \sum_{r \in \mathbb{R}^n} \Psi \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} s_r + \Psi \begin{pmatrix} p_1^* \\ 1 \\ 0 \end{pmatrix} y_1 + \Psi \begin{pmatrix} p_2^* \\ 0 \\ 1 \end{pmatrix} y_2 \\
&\geq 1.
\end{aligned}$$

Hence (A.3) is valid for (A.1).  $\square$

Using Theorem 5 with  $\mathcal{R} = \mathbb{R}^n$  and  $\mathcal{P} = \{p_1^*, p_2^*\}$ , one can show that  $V_\psi(p_2^*; p_1^*)$  is the infimum over all  $\lambda > 0$  such that (A.3) is valid for (A.1). Hence, Claims 30 and 31 give the desired result.  $\square$

**A.3. Proof of Theorem 21.** We state relevant results, before giving the final proof of Theorem 21 at the end of the section.

The first result is an extension of the so-called ‘Collision Lemma’ (Lemma 3.2 in [9]). Recall the definition of  $B(V_\psi(p^*), p^*)$  in (4.2).

PROPOSITION 32. Let  $B \subseteq \mathbb{R}^n$  be any maximal  $S$ -free 0-neighborhood of the form  $B = \{x \in \mathbb{R}^n : a^i \cdot x \leq 1, i \in I\}$  and let  $p^* \in \mathbb{R}^n$ . Let  $(\bar{x}, \bar{x}_{n+1}), (\bar{y}, \bar{y}_{n+1}) \in B(V_\psi(p^*), p^*) \cap (S \times \mathbb{Z}_+)$ , and  $i_x, i_y \in I$  be such that  $(a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*) \cdot (\bar{x}, \bar{x}_{n+1}) = 1$  and  $(a^{i_y}, V_\psi(p^*) - a^{i_y} \cdot p^*) \cdot (\bar{y}, \bar{y}_{n+1}) = 1$ . Let  $(x, k_x) \in R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1})$  and  $(y, k_y) \in R_{B(V_\psi(p^*), p^*)}(\bar{y}, \bar{y}_{n+1})$  with  $k_x, k_y \in \mathbb{Z}$ ,  $0 \leq k_x \leq \bar{x}_{n+1}$ , and  $0 \leq k_y \leq \bar{y}_{n+1}$ . If  $x - y \in W_S$  then

$$(a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*) \cdot (x, k_x) = (a^{i_y}, V_\psi(p^*) - a^{i_y} \cdot p^*) \cdot (y, k_y).$$

Furthermore, if it also holds that  $(x, k_x) \in \text{int}(R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1}))$  and  $(y, k_y) \in \text{int}(R_{B(V_\psi(p^*), p^*)}(\bar{y}, \bar{y}_{n+1}))$  then  $(a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*) = (a^{i_y}, V_\psi(p^*) - a^{i_y} \cdot p^*)$ .

*Proof.* Let  $(x, k_x) \in R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1})$  and  $(y, k_y) \in R_{B(V_\psi(p^*), p^*)}(\bar{y}, \bar{y}_{n+1})$ . Assume to the contrary that  $(a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*) \cdot (x, k_x) < (a^{i_y}, V_\psi(p^*) - a^{i_y} \cdot p^*) \cdot (y, k_y)$  and consider  $(\bar{y}, \bar{y}_{n+1}) + (x - y, k_x - k_y)$  (if the inequality is reversed then consider  $(\bar{x}, \bar{x}_{n+1}) + (y - x, k_y - k_x)$  instead). Since  $x - y \in W_S$  and  $k_y \leq \bar{y}_{n+1}$ , it follows that  $(z, z_{n+1}) := (\bar{y}, \bar{y}_{n+1}) + (x - y, k_x - k_y) = (\bar{y} + (x - y), (\bar{y}_{n+1} - k_y) + k_x) \in S \times \mathbb{Z}_+$ . We claim that  $(z, z_{n+1}) \in \text{int}(B(V_\psi(p^*), p^*))$ , contradicting that  $B(V_\psi(p^*), p^*)$  is  $S \times \mathbb{Z}_+$  free. We will show this using the halfspace definition of  $B(V_\psi(p^*), p^*)$  from (4.2). In what follows, for  $i \in I$  define

$$\alpha_i := (a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*).$$

Take  $i \in I$ . If  $i = i^x$ , it follows that

$$\begin{aligned} \alpha_{i^x} \cdot (z, z_{n+1}) &\leq 1 - \alpha_{i^x}(y, k_y) + \alpha_{i^x} \cdot (x, k_x) && \text{since } (\bar{y}, \bar{y}_{n+1}) \in S \times \mathbb{Z}_+ \\ &< 1 - \alpha_{i^y}(y, k_y) + \alpha_{i^x} \cdot (x, k_x) && \text{since } (y, k_y) \in R_{B(V_\psi(p^*), p^*)}(\bar{y}, \bar{y}_{n+1}) \\ &\leq 1 && \text{since } a_{i^x} \cdot (x, k_x) < a_{i^y} \cdot (y, k_y). \end{aligned}$$

If  $i = i^y$ , it follows that

$$\begin{aligned} \alpha_{i^y} \cdot (z, z_{n+1}) &= 1 - \alpha_{i^y}(y, k_y) + \alpha_{i^y} \cdot (x, k_x) && \text{since } (\bar{y}, \bar{y}_{n+1}) \in S \times \mathbb{Z}_+ \\ &< 1 - \alpha_{i^x}(x, k_x) + \alpha_{i^y} \cdot (x, k_x) && \text{since } a_{i^x} \cdot (x, k_x) < a_{i^y} \cdot (y, k_y) \\ &= 1. \end{aligned}$$

Finally, if  $i \in I \setminus \{i^x, i^y\}$  then

$$\begin{aligned} \alpha_i \cdot (z, z_{n+1}) &\leq 1 + \alpha_i \cdot (x, k_x) - \alpha_i \cdot (y, k_y) && \text{since } (\bar{y}, \bar{y}_{n+1}) \in S \times \mathbb{Z}_+ \\ &\leq 1 + \alpha_i \cdot (x, k_x) - \alpha_{i^y} \cdot (y, k_y) && \text{since } (y, k_y) \in R_{B(V_\psi(p^*), p^*)}(\bar{y}, \bar{y}_{n+1}) \\ &< 1 + \alpha_i \cdot (x, k_x) - \alpha_{i^x} \cdot (x, k_x) && \text{since } a_{i^x} \cdot (x, k_x) < a_{i^y} \cdot (y, k_y) \\ &< 1 && \text{since } (x, k_x) \in R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1}). \end{aligned}$$

Hence  $(z, z_{n+1}) \in \text{int}(B(V_\psi(p^*), p^*))$  giving the desired contradiction.

Now suppose the containments  $(x, k_x) \in \text{int}(R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1}))$  and  $(y, k_y) \in \text{int}(R_{B(V_\psi(p^*), p^*)}(\bar{y}, \bar{y}_{n+1}))$  both hold. Assume to the contrary that  $\alpha_{i^x} \neq \alpha_{i^y}$ . We will again show  $(z, z_{n+1}) \in \text{int}(B(V_\psi(p^*), p^*))$ . Since  $\alpha_{i^x} \neq \alpha_{i^y}$  and  $(y, k_y) \in \text{int}(R_{B(V_\psi(p^*), p^*)}(\bar{y}, \bar{y}_{n+1}))$ , it follows that

$$(A.4) \quad \alpha_{i^x} \cdot (y, k_y) < \alpha_{i^y} \cdot (y, k_y)$$

and

$$(A.5) \quad \alpha_{i^x} \cdot (\bar{y} - y, \bar{y}_{n+1} - k_y) < \alpha_{i^x} \cdot (\bar{y} - y, \bar{y}_{n+1} - k_y)$$

From the previous argument that  $\alpha_{i^x}(x, k_x) = \alpha_{i^y}(y, k_y)$ . Take  $i \in I$ . If  $i = i^x$  then

$$\begin{aligned} \alpha_{i^x} \cdot (z, z_{n+1}) &= \alpha_{i^x} \cdot (\bar{y} - y, \bar{y}_{n+1} - k_y) + \alpha_{i^x} \cdot (x, k_x) \\ &< \alpha_{i^y} \cdot (\bar{y} - y, \bar{y}_{n+1} - k_y) + \alpha_{i^x} \cdot (x, k_x) && \text{from (A.5)} \\ &= 1 - \alpha_{i^y}(y, k_y) + \alpha_{i^x} \cdot (x, k_x) && \text{since } (\bar{y}, \bar{y}_{n+1}) \in S \times \mathbb{Z}_+ \\ &= 1. \end{aligned}$$

If  $i = i^y$ , it follows that

$$\begin{aligned} \alpha_{iy} \cdot (z, z_{n+1}) &= 1 - \alpha_{iy}(y, k_y) + \alpha_{iy} \cdot (x, k_x) \quad \text{since } (\bar{y}, \bar{y}_{n+1}) \in S \times \mathbb{Z}_+ \\ &= 1 - \alpha_{ix}(x, k_x) + \alpha_{iy} \cdot (x, k_x) \\ &< 1 \quad \text{since } (x, k_x) \in R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1}). \end{aligned}$$

Finally, if  $i \in I \setminus \{i^x, i^y\}$  then

$$\begin{aligned} &\alpha_i \cdot (z, z_{n+1}) \\ &= \alpha_i \cdot (\bar{y} - y, \bar{y}_{n+1} - k_y) + \alpha_i \cdot (x, k_x) \\ &< \alpha_{iy} \cdot (\bar{y} - y, \bar{y}_{n+1} - k_y) + \alpha_i \cdot (x, k_x) \quad \text{since } (y, k_y) \in R_{B(V_\psi(p^*), p^*)}(\bar{y}, \bar{y}_{n+1}) \\ &< 1 - \alpha_{iy}(y, k_y) + \alpha_{ix} \cdot (x, k_x) \quad \text{since } (x, k_x) \in R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1}) \\ &= 1. \end{aligned}$$

□

LEMMA 33. [Theorem 9.4 in [19]] Let  $P_\omega \subseteq \mathbb{R}^n, \omega \in \Omega$  be a (possibly infinite) family of polyhedra such that any bounded set intersects only finitely many polyhedra, and  $\bigcup_{\omega \in \Omega} P_\omega = \mathbb{R}^n$ . Suppose there is a family of functions  $A_\omega : P_\omega \rightarrow \mathbb{R}^n, \omega \in \Omega$  such that  $A_\omega$  is continuous over  $P_\omega$  for each  $\omega \in \Omega$ , and for every pair  $\omega_1, \omega_2 \in \Omega$ ,  $A_{\omega_1}(x) = A_{\omega_2}(x)$  for all  $x \in P_{\omega_1} \cap P_{\omega_2}$ . Then there is a unique, continuous map  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  that equals  $A_\omega$  when restricted to  $P_\omega$  for each  $\omega \in \Omega$ .

*Proof.* This follows from a direct application of Theorem 9.4 in Chapter III of [19] by noting that polyhedra are closed sets. □

PROPOSITION 34. Let  $B$  be a maximal  $S$ -free convex 0-neighborhood such that  $\text{int}(B \cap \text{conv}(S)) \neq \emptyset$ . Then any bounded set  $U \subseteq \mathbb{R}^n$  intersects a finite number of polyhedra from  $\mathcal{X}(B, p^*) + W_S$ .

*Proof.* Recall that  $B \subseteq \mathbb{R}^n$  is a full dimensional set, and by construction, so is  $B(V_\psi(p^*), p^*)$ . Furthermore,  $\text{int}(\text{conv}(S) \cap B) \neq \emptyset$  and therefore,  $\text{int}(\text{conv}(S \times \mathbb{Z}_+) \cap B(V_\psi(p^*), p^*)) \neq \emptyset$ . Define  $\tilde{U} := U \times [0, 1] \subseteq \mathbb{R}^{n+1}$ . Note that  $\tilde{U}$  is bounded in  $\mathbb{R}^{n+1}$ , and using Theorem 2.7 in [9],  $\tilde{U}$  intersects finitely many polyhedra from  $R(B(V_\psi(p^*), p^*)) + W_{S \times \mathbb{Z}_+} = R(B(V_\psi(p^*), p^*)) + W_S \times \{0\}$ . Say for  $i = 1, \dots, k$ ,  $\tilde{U}$  intersects  $\tilde{P}_i + (w_i, 0)$ ,  $(w_i, 0) \in W_S \times \{0\}$  and  $\tilde{P}_i$  is a polyhedron in  $R(B(V_\psi(p^*), p^*))$ .

For any  $t \in \mathbb{Z}$ , Proposition 16 states that the projection of  $H_t \cap (\tilde{P}_i + (w_i, 0))$  onto  $\mathbb{R}^n$  is  $(H_0 \cap \tilde{P}_i)|_{\mathbb{R}^n} + tp^* + w_i$ , where  $\cdot|_{\mathbb{R}^n}$  denotes the projection onto the first  $n$  coordinates. By definition of  $\mathcal{X}(B, p^*) + W_S$ , all polyhedra in  $\mathcal{X}(B, p^*) + W_S$  are of the form  $(H_0 \cap \tilde{P}_i)|_{\mathbb{R}^n} + tp^* + w_i$  for some  $t$  less than a blocking point corresponding to  $B(V_\psi(p^*), p^*)$ . Notice that, since  $\tilde{U}$  is bounded,  $H_t \cap \tilde{U} \cap (\tilde{P}_i + (w_i, 0)) \neq \emptyset$  for only a finite number of integral  $t$ , for each  $i = 1, \dots, k$ . Hence  $U$  only intersects a finite number of polyhedra from  $\mathcal{X}(B, p^*)$ . □

PROPOSITION 35. Let  $B$  be a maximal  $S$ -free convex 0-neighborhood. For  $p^* \in \mathbb{R}^n$ , the set  $\mathcal{X}(B, p^*) + W_S$  is closed.

*Proof.* Let  $x \notin \mathcal{X}(B, p^*) + W_S$  and consider the open ball  $D(x, 1)$  around  $x$  of radius 1. From Proposition 34,  $D(x, 1)$  intersects only finite many polyhedra  $P_1, \dots, P_k$  from  $\mathcal{X}(B, p^*) + W_S$ . Since each  $P_i$  is closed, so is the finite union  $\bigcup_{i=1}^k P_i$ . Since  $x \notin \bigcup_{i=1}^k P_i$ , there exists  $\varepsilon > 0$  such that the open ball  $D(x; \varepsilon) \subseteq D(x; 1)$  does not intersect  $P_i$ , for  $i = 1, \dots, k$ . Therefore,  $D(x; \varepsilon) \cap (\mathcal{X}(B, p^*) + W_S) = \emptyset$ . This implies that  $\mathbb{R}^n \setminus (\mathcal{X}(B, p^*) + W_S)$  is open, and thus  $\mathcal{X}(B, p^*) + W_S$  is closed. □

Let  $m$  be as in Theorem 21. For each  $i \in I$ , define  $a_m^i := \frac{a^i}{1+a^i \cdot m}$ . Note that

$$B + m = \{r \in \mathbb{R}^n : a_m^i \cdot r \leq 1, \forall i \in I\}$$

and

$$\begin{aligned} & B(V_\psi(p^*), p^*) + (m, 0) \\ &= \left\{ (r, r_{n+1}) \in \mathbb{R}^{n+1} : a_m^i \cdot r + \left( \frac{V_\psi(p^*) - a^i \cdot p^*}{1 + a^i \cdot m} \right) r_{n+1} \leq 1, \forall i \in I \right\}. \end{aligned}$$

Note the apex of  $B(V_\psi(p^*), p^*) + (m, 0)$  is  $\frac{1}{V_\psi(p^*)}(p^* + V_\psi(p^*)m, 1)$ . Define

$$(A.6) \quad \hat{p} := p^* + V_\psi(p^*)m.$$

For each  $k \in \mathbb{Z}$ ,  $k \geq 0$ , and  $i \in I$  define  $T_i^k : \mathbb{R}^n \rightarrow \mathbb{R}^n$  to be

$$T_i^k(x) := x + (a^i, V_\psi(p^*) - a^i p^*) \cdot (x, k)m.$$

Using a direct calculation, the map  $T_i^k$  is seen to be invertible.

PROPOSITION 36. *The function  $T_i^k$  is invertible with the inverse defined by*

$$(T_i^k)^{-1}(x) = x - \left( a_m^i, \frac{V_\psi(p^*) - a^i \cdot p^*}{1 + a^i \cdot m} \right) \cdot (x, k)m.$$

LEMMA 37. *Let  $(\bar{x}, \bar{x}_{n+1}), (\bar{y}, \bar{y}_{n+1}) \in B(V_\psi(p^*), p^*) \cap (S \times \mathbb{Z}_+)$  and take  $i_x, i_y \in I$  to be such that  $(a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*) \cdot (\bar{x}, \bar{x}_{n+1}) = 1$  and  $(a^{i_y}, V_\psi(p^*) - a^{i_y} \cdot p^*) \cdot (\bar{y}, \bar{y}_{n+1}) = 1$ . Assume  $(z, k_x) \in R_{B(V_\psi(p^*), p^*)}(\bar{x}, \bar{x}_{n+1}) + (w_x, 0)$  and  $(z, k_y) \in R_{B(V_\psi(p^*), p^*)}(\bar{y}, \bar{y}_{n+1}) + (w_y, 0)$ , where  $w_x, w_y \in W_S$ ,  $k_i \in \mathbb{Z}_+$ ,  $k_x \leq \bar{x}_{n+1}$ , and  $k_y \leq \bar{y}_{n+1}$ . Then  $T_{i_x}^{k_x}(z - w_x, k_x) + w_x = T_{i_y}^{k_y}(z - w_y, k_y) + w_y$ .*

*Proof.* A direct calculation shows that

$$\begin{aligned} & T_{i_x}^{k_x}(z - w_x, k_x) + w_x \\ &= (z - w_x) + (a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*) \cdot (z, k_x)m + w_x && \text{by definition} \\ &= z + (a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*) \cdot (z, k_x)m \\ &= z + (a^{i_y}, V_\psi(p^*) - a^{i_y} \cdot p^*) \cdot (z, k_y)m && \text{by Proposition 32} \\ &= (z - w_y) + (a^{i_y}, V_\psi(p^*) - a^{i_y} \cdot p^*) \cdot (z, k_y)m + w_y && \text{by definition} \\ &= T_{i_y}^{k_y}(z - w_y, k_y) + w_y \end{aligned}$$

□

PROPOSITION 38. *Let  $(\bar{x}, \bar{x}_{n+1}) \in B(V_\psi(p^*), p^*)$ . Consider  $R_B(\bar{x} - \bar{x}_{n+1}p^*) + kp^*$  for  $k \in \mathbb{Z}_+$ ,  $k \leq \bar{x}_{n+1}$ . Let  $i_x \in I$  be such that  $(a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*) \cdot (\bar{x}, \bar{x}_{n+1}) = 1$ . Then*

$$T_{i_x}^k(R_B(\bar{x} - \bar{x}_{n+1}p^*) + kp^*) = R_{B+m}(\bar{x} + m - \bar{x}_{n+1}\hat{p}) + k\hat{p},$$

where  $\hat{p}$  is defined in (A.6).

*Proof.* Let  $y \in R_B(\bar{x} - \bar{x}_{n+1}p^*) + kp^*$ . Note that  $(y, k) \in R_{B(V_\psi(p^*), p^*)}((\bar{x}, \bar{x}_{n+1}))$  by Proposition 16. Also,  $T_{i_x}^k(y) \in R_{B+m}(\bar{x} + m - \bar{x}_{n+1}\hat{p}) + k\hat{p}$  if and only if  $(T_{i_x}^k(y), k) \in R_{B(V_\psi(p^*), p^*) + (m, 0)}((\bar{x} + m, \bar{x}_{n+1}))$ ; we will show this sufficient condition.

We first show that for any  $i \in I$  such that  $[(a^i, V_\psi(p^*) - a^i \cdot p^*) - (a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*)] \cdot (y, k) \leq 0$ , it follows that  $\left(a_m^i, \frac{V_\psi(p^*) - a^i \cdot p^*}{1 + a^i \cdot m}\right) \cdot (T_{i_x}^k(y), k) \leq a^{i_x} \cdot y + k(V_\psi(p^*) - a^{i_x} \cdot p^*)$  with equality for  $i = i_x$ . Indeed, direct calculation shows that

$$\begin{aligned} & \left(a_m^i, \frac{V_\psi(p^*) - a^i \cdot p^*}{1 + a^i \cdot m}\right) \cdot (T_{i_x}^k(y), k) \\ &= \left(\frac{a^i}{1 + a^i \cdot m}, \frac{V_\psi(p^*) - a^i \cdot p^*}{1 + a^i \cdot m}\right) \cdot (y + (a^{i_x} \cdot y + (V_\psi(p^*) - a^{i_x} \cdot p^*)k)m, k) \\ &= \frac{a^i \cdot y + k(V_\psi(p^*) - a^i \cdot p^*) + (a^{i_x} \cdot y)(a^i \cdot m) + (a^i \cdot m)k(V_\psi(p^*) - a^{i_x} \cdot p^*)}{1 + a^i \cdot m} \\ &\leq \frac{a^{i_x} \cdot y + k(V_\psi(p^*) - a^{i_x} \cdot p^*) + (a^{i_x} \cdot y)(a^i \cdot m) + (a^i \cdot m)k(V_\psi(p^*) - a^{i_x} \cdot p^*)}{1 + a^i \cdot m} \\ &= \frac{(1 + a^i \cdot m)(a^{i_x} \cdot y + k(V_\psi(p^*) - a^{i_x} \cdot p^*))}{1 + a^i \cdot m} \\ &= a^{i_x} \cdot y + k(V_\psi(p^*) - a^{i_x} \cdot p^*), \end{aligned}$$

where the inequality arises since  $[(a^i, V_\psi(p^*) - a^i \cdot p^*) - (a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*)] \cdot (y, k) \leq 0$ . Note that equality holds when  $i = i_x$ .

Using a similar argument, it follows that for any  $i \in I$  such that  $[(a^i, V_\psi(p^*) - a^i \cdot p^*) - (a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*)] \cdot (y, k) \leq 0$ , it follows that  $\left(a_m^i, \frac{V_\psi(p^*) - a^i \cdot p^*}{1 + a^i \cdot m}\right) \cdot (\bar{x} + m - T_{i_x}^k(y), \bar{x}_{n+1} - k) \leq 1 - (a^{i_x} \cdot y + (V_\psi(p^*) - a^{i_x} \cdot p^*)k)$  with equality for  $i = i_x$ .

Since  $(y, k) \in R_{B(V_\psi(p^*), p^*)}((\bar{x}, \bar{x}_{n+1}))$ , it follows that  $[(a^i, V_\psi(p^*) - a^i \cdot p^*) - (a^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*)] \cdot (y, k) \leq 0$  for each  $i \in I$ . Applying the arguments to each  $i \in I$ , with equality for  $i = i_x$ , we see that

$$\left[\left(a_m^i, \frac{V_\psi(p^*) - a^i \cdot p^*}{1 + a^i \cdot m}\right) - \left(a_m^{i_x}, \frac{V_\psi(p^*) - a^{i_x} \cdot p^*}{1 + a^{i_x} \cdot m}\right)\right] \cdot (T_{i_x}^k(y), k) \leq 0$$

and

$$\left[\left(a_m^i, \frac{V_\psi(p^*) - a^i \cdot p^*}{1 + a^i \cdot m}\right) - \left(a_m^{i_x}, \frac{V_\psi(p^*) - a^{i_x} \cdot p^*}{1 + a^{i_x} \cdot m}\right)\right] \cdot (\bar{x}_{n+1} + m - T_{i_x}^k(y), \bar{x}_{n+1} - k) \leq 0$$

for each  $i \in I$ . Hence  $(T_{i_x}^k(y), k) \in R_{B(V_\psi(p^*), p^*) + (m, 0)}((\bar{x} + m, \bar{x}_{n+1}))$  and so

$$T_{i_x}^k(R_B(\bar{x} - \bar{x}_{n+1}p^*) + kp^*) \subseteq R_{B+m}(\bar{x} + m - \bar{x}_{n+1}\hat{p}) + k\hat{p}.$$

Using similar reasoning applied to  $(T_{i_x}^k)^{-1}$ , we get the reverse inclusion.  $\square$

*Proof of Theorem 21.* Recall the definition of  $\hat{p}$  in (A.6). We will first show that if  $\mathcal{X}(B, p^*) + W_S = \mathbb{R}^n$  then  $\mathcal{X}(B + m, \hat{p}) + W_{S+m} = \mathbb{R}^n$ . The converse holds by switching the roles of  $(B, p^*)$  and  $(B + m, \hat{p})$ .

Note that a direct calculation shows that  $W_S = W_{S+m}$  (see Proposition 2.1 in [9]). If  $B$  is a half-space, then the lifting region is equal to  $\mathbb{R}^n$ . Note the lifting region is contained in  $\mathcal{X}(B, p^*) + W_S$ , and therefore  $\mathcal{X}(B, p^*) + W_S = \mathcal{X}(B + m, \hat{p}) + W_{S+m} = \mathbb{R}^n$ . So assume that  $B$  is not a half-space.

Define the map  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  to be

$$A(y) := T_{i_x}^k(y - u) + u, \quad \text{if } y \in R_B(w(z)) + kp^* + u,$$

where  $z = (\bar{x}, \bar{x}_{n+1})$  is a blocking point of  $B(V_\psi(p^*), p^*)$ ,  $k \in \mathbb{Z}_+$ ,  $k \leq \bar{x}_{n+1}$ ,  $u \in W_S$ , and  $(a_m^{i_x}, V_\psi(p^*) - a^{i_x} \cdot p^*) \cdot (\bar{x}, \bar{x}_{n+1}) = 1$ . Since  $\mathcal{X}(B, p^*) + W_S = \mathbb{R}^n$ , each  $y$  is in some  $R_B(\bar{x} - \bar{x}_{n+1}p^*) + kp^* + u$ . Moreover,  $A$  is well defined from Lemma 37.

By assumption,  $\mathbb{R}^n = \mathcal{X}(B, p^*) + W_S = \mathbb{R}^n$ . Therefore,

$$\begin{aligned} A(\mathbb{R}^n) &= A(\mathcal{X}(B, p^*) + W_S) \\ &= A\left(\bigcup_{(\bar{x}, \bar{x}_{n+1}) \in B(V_\psi(p^*), p^*) \cap (S \times \mathbb{Z}_+), u \in W_S} \left(\bigcup_{i=0}^{\bar{x}_{n+1}} (R_B(\bar{x} - \bar{x}_{n+1}p^*) + ip^* + u)\right)\right) \\ &= \bigcup_{(\bar{x}, \bar{x}_{n+1}) \in B(V_\psi(p^*), p^*) \cap (S \times \mathbb{Z}_+), u \in W_S} \left(\bigcup_{i=0}^{\bar{x}_{n+1}} A(R_B(\bar{x} - \bar{x}_{n+1}p^*) + ip^* + u)\right) \\ &= \bigcup_{(\bar{x}, \bar{x}_{n+1}) \in B(V_\psi(p^*), p^*) \cap (S \times \mathbb{Z}_+), u \in W_S} \left(\bigcup_{i=0}^{\bar{x}_{n+1}} R_{B+m}(\bar{x} + m - \bar{x}_{n+1}\hat{p}) + i\hat{p} + u\right) \\ &= \left(\bigcup_{(\bar{x}, \bar{x}_{n+1}) \in B(V_\psi(p^*), p^*) \cap (S \times \mathbb{Z}_+)} \left(\bigcup_{i=0}^{\bar{x}_{n+1}} R_{B+m}(\bar{x} + m - \bar{x}_{n+1}\hat{p}) + i\hat{p}\right)\right) + W_{S+m} \\ &= \mathcal{X}(B + m, \hat{p}) + W_{S+m}. \end{aligned}$$

The fourth equality follows from Proposition 38. Hence,  $A$  maps the translated fixing region to the translated fixing region.

For the time being, suppose that  $A$  is injective. From Lemma 33 and Proposition 34,  $A$  is continuous. Therefore, the Invariance of Domain Theorem (see [10, 18]) states that  $A$  is an open map. As  $A$  maps  $\mathbb{R}^n$  to the translated fixing region, the translated fixing region is open. From Proposition 35, the translated fixing region is closed, and so the translated fixing region is both open and closed. As the translated fixing region is non-empty, this implies that it must be  $\mathbb{R}^n$ , and so  $B + m$  is also one point fixable. Thus it is sufficient to show that  $A$  is injective.

Suppose that  $A(y_1) = A(y_2)$  for some  $y_1, y_2 \in \mathbb{R}^n$ . Let  $\alpha := A(y_1) = A(y_2)$ . By definition, for  $j = 1, 2$ , there exists a blocking point  $(\bar{x}^j, \bar{x}_{n+1}^j) \in S \times \mathbb{Z}_+$ ,  $k_j \in \mathbb{Z}_+$  with  $k_j \leq \bar{x}_{n+1}^j$ , and  $w_j \in W_S$  such that  $y_j \in R_B(\bar{x}^j - \bar{x}_{n+1}^j p^*) + k_j p^* + w_j$ . Moreover

$$\alpha = A(y_1) = T_{i_{x_1}}^{k_1}(y_1 - w_1) + w_1 = T_{i_{x_2}}^{k_2}(y_2 - w_2) + w_2 = A(y_2).$$

From Proposition 38,  $\alpha \in R_{B+m}(\bar{x}^j + m - \bar{x}_{n+1}^j \hat{p}) + k_j \hat{p} + w_j$ , for  $j \in \{1, 2\}$ . Hence  $(\alpha, k_j) \in R_{B(V_\psi(p^*), p^*) + (m, 0)}((\bar{x}^j + m, \bar{x}_{n+1}^j)) + (w_j, 0)$ , for  $j \in \{1, 2\}$ . According to Lemma 37 applied to  $(T_{i_{x_1}}^{k_1})^{-1}$  and  $(T_{i_{x_2}}^{k_2})^{-1}$ , we see that

$$(T_{i_{x_1}}^{k_1})^{-1} \left( T_{i_{x_1}}^{k_1}(y_1 - w_1) + w_1 - w_1 \right) + w_1 = (T_{i_{x_2}}^{k_2})^{-1} \left( T_{i_{x_2}}^{k_2}(y_2 - w_2) + w_2 - w_2 \right) + w_2.$$

Applying the definition of each  $(T_{i_{x_j}}^{k_j})^{-1}$  for  $j = 1, 2$  and simplifying the results, we see that  $y_1 = y_2$ . Hence  $A$  is injective.  $\square$

## Appendix B. Case Analysis for $K_i$ from Claim 28.

*Proof of Claim 28.* To prove this claim, we first construct the half-space definition of the spindles  $R_T(s_4 - p^*)$ ,  $R_T(s_5 - p^*)$ , and  $R_T(s_6 - 2p^*)$ . For sake of presentation, consider the following vectors

$$\begin{aligned} q_1 &= \left( \frac{1}{(1, \gamma_1) \cdot (b_1 + 1, b_2 + 1)}, \frac{\gamma_1}{(1, \gamma_1) \cdot (b_1 + 1, b_2 + 1)} \right) \\ q_2 &= \left( -\frac{1}{(-1, \gamma_2) \cdot (b_1, b_2 + 1)}, \frac{\gamma_2}{(-1, \gamma_2) \cdot (b_1, b_2 + 1)} \right) \\ q_3 &= \left( \frac{\gamma_3}{(\gamma_3, -1) \cdot (b_1, b_2)}, -\frac{1}{(\gamma_3, -1) \cdot (b_1, b_2)} \right) \end{aligned}$$

defining  $T$  (see (5.4)). Since  $(s_4, z_4) = (s_4, 1) \in P$  is contained in the same facet as  $(s_1, 0)$  (see the discussion following (5.6)), we see that

$$(B.1) \quad R_T(s_4 - p^*) = \{x \in \mathbb{R}^2 : (q_i - q_1) \cdot x \leq 0, (q_i - q_1) \cdot (s_4 - p^* - x) \leq 0, i \in \{2, 3\}\}.$$

Similarly, as  $(s_5, 1), (s_6, 2)$  share a facet with  $(s_2, 0)$  and  $(s_3, 0)$ , respectively, we see

$$(B.2)$$

$$R_T(s_5 - p^*) = \{x \in \mathbb{R}^2 : (q_i - q_2) \cdot x \leq 0, (q_i - q_2) \cdot (s_5 - p^* - x) \leq 0, i \in \{1, 3\}\}$$

$$(B.3)$$

$$R_T(s_6 - 2p^*) = \{x \in \mathbb{R}^2 : (q_i - q_3) \cdot x \leq 0, (q_i - q_3) \cdot (s_6 - 2p^* - x) \leq 0, i \in \{1, 2\}\}.$$

Consider the collection of points  $K_1 = \text{conv}\{l, e_1, g, u_0\}$ . In order to prove  $K_1 \subseteq R_T(s_4 - p^*)$ , it is enough to show that  $\{l, e_1, g, u_0\} \subseteq R_T(s_4 - p^*)$ . Consider the point  $l \in \{l, e_1, g, u_0\}$ . Using the values in Figure 1, (B.1), and the definition  $s_4 = (1 + b_1, 2 + b_2)$ , it is straight forward, yet tedious, to show that the four values  $(q_i - q_1) \cdot l, (q_i - q_1) \cdot (s_4 - p^* - l), i \in \{2, 3\}$  are all contained in the set

$$(B.4)$$

$$Q = \left\{ \begin{array}{lll} 0, & \frac{-1}{(1, \gamma_1) \cdot (1 + b_1, b + b_2)}, & \frac{-\gamma_1}{(1, \gamma_1) \cdot (1 + b_1, b + b_2)}, & \frac{-1 + \gamma_2}{(-1, \gamma_2) \cdot (b_1, b_2)}, \\ & \frac{-2 + \gamma_3}{(-2\gamma_3, -2) \cdot (b_1, b_2)}, & \frac{\gamma_3}{(-2\gamma_3, -2) \cdot (b_1, b_2)}, & \frac{-1 + \gamma_3}{(\gamma_3, -1) \cdot (b_1, b_2)}, \\ & \frac{-1 - \gamma_1}{(1, \gamma_1) \cdot (1 + b_1, 1 + b_2)}, & \frac{-b_1(1 + \gamma_1\gamma_3) - (1 + \gamma_1)}{(1 + b_1 + \gamma_1(1 + b_2))(-b_2 + b_1\gamma_3)}, & \frac{b_1(-1 + \gamma_3) + b_2(-1 + \gamma_2) + \gamma_2}{(b_1 - (1 + b_2)\gamma_2)(-b_2 + b_1\gamma_3)}, \\ & \frac{-(b_1 + 1) + \gamma_1 b_2 - (\gamma_1 + 2\gamma_1^2\gamma_3)}{(1 + b_1 + \gamma_1(1 + b_2))(-b_2 + b_1\gamma_3)}, & \frac{\gamma_2 + b_1(-1 + \gamma_2\gamma_3)}{(b_1 - (1 + b_2)\gamma_2)(-b_2 + b_1\gamma_3)} \end{array} \right\}.$$

Using the assumptions  $\gamma_1, \gamma_2, \gamma_3 > 0$ ,  $\gamma_2, \gamma_3 < 1$ , and  $-1 \leq b_2 \leq b_1 \leq 0$ , a direct calculation shows that every value in  $Q$  is nonpositive. Hence, from (B.1),  $l \in R_T(s_4 - p^*)$ . Similar arguments show that when the four inner products defining (B.1) are evaluated at any point in  $\{l, e_1, g, u_0\}$ , the result is in  $Q$ . Hence  $\{l, e_1, g, u_0\} \subseteq R_T(s_4 - p^*)$ .

Next we show that  $K_2 \subseteq R_T(s_5 - p^*) + (1, 1)$ . For this, it is enough to show that  $\{u_0 - (1, 1), m - (1, 1), i - (1, 1), g - (1, 1)\} \subseteq R_T(s_5 - p^*)$ . However, substituting these four values into the expressions in (B.2), yields numbers in  $Q$ . Hence  $\{u_0 - (1, 1), m - (1, 1), i - (1, 1), g - (1, 1)\} \subseteq R_T(s_5 - p^*)$ .

In order to show  $K_3 \subseteq R_T(s_4 - p^*) + p^*$ ,  $K_4 \subseteq R_T(s_5 - p^*) + p^*$ , and  $K_5 \subseteq R_T(s_6 - 2p^*) + p^*$ , we can use an argument similar to that used in proving  $K_2 \subseteq R_T(s_5 - p^*) + (1, 1)$  (all of the inner products evaluated are contained in  $Q$ ).  $\square$